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Fourier–Mukai and Nahm transforms for holomorphic triples on elliptic curves

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Abstract

We define a Fourier–Mukai transform for a triple consisting of two holomorphic vector bundles over an elliptic curve and a homomorphism between them. We prove that in some cases, the transform preserves the natural stability condition for a triple. We also define a Nahm transform for solutions to natural gauge-theoretic equations on a triple—vortices—and explore some of its basic properties. Our approach combines direct methods with dimensional reduction techniques, relating triples over a curve with vector bundles over the product of the curve with the complex projective line.
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1. Introduction

The Fourier–Mukai transform, as originally introduced by Mukai for abelian varieties [22] establishes a duality between the derived categories of coherent sheaves over an abelian variety and its dual variety. The theory has been extended to more general varieties, including K3 surfaces [3], Calabi–Yau threefolds or elliptic fibrations [9,16]. In particular, it is a very powerful tool in the study of moduli spaces of vector bundles over abelian surfaces and K3 surfaces (see [9,16,21,23,28] for instance). In the gauge-theoretic side, the Nahm transform provides a differential geometric analogue of the Fourier–Mukai transform relating instantons (or monopoles) on dual manifolds [4,6,17,24]. In many cases, whenever it makes sense, both transforms are compatible in a suitable way.

In this paper, we study Fourier–Mukai and Nahm transforms for holomorphic triples over an elliptic curve and their corresponding vortex equations. A triple here consists of two holomorphic vector bundles over the elliptic curve and a homomorphism between them. The motivation to study this problem is two-fold. On the one hand the Nahm transform has been successfully applied to find instanton and monopole solutions, which are defined in real dimensions 4 and 3, respectively. It is then very natural to try to find an analogue for two-dimensional vortices. On the other hand, vortices in two dimensions are equivalent to $SU(2)$-invariant instantons over the product of the elliptic curve and the Riemann sphere, where the $SU(2)$ action is given simply by the usual one on the sphere. This suggests a relative four-dimensional approach to the problem. In a related context, the Nahm transform has been successfully applied to study doubly periodic instantons and their relationship with Hitchin's equations [18,19].

Here is a description of the paper. In Section 2, we briefly review the Fourier–Mukai and Nahm transforms for vector bundles over elliptic curves. We recall the preservation of stability and prove that the constant central curvature condition for a connection (which on a curve coincides with the Einstein–Hermitian condition) is preserved. Although the latter seems to be of general knowledge, we have not found it in the literature and hence include it here since it is relevant for our further study for triples. We follow the approach given in [13].

In Section 3, we review first the basic stability theory for triples. An important feature is that the stability criterium depends on a real parameter which is typically bounded [7]. We then introduce the Fourier–Mukai transform for triples on elliptic curves and give two natural approaches for transforming a triple. The first one is based on the absolute Fourier–Mukai transform acting on the components of the triple. The second approach is based on a relative Fourier–Mukai transform combined with a dimensional reduction procedure. We prove that the Fourier–Mukai transform preserves stability of triples for “small” and “large” values of the stability parameter, providing an isomorphism of moduli spaces. What happens for other values of the parameter remains to be investigated. We conclude this section by applying these results to obtain isomorphisms between moduli spaces of stable $SU(2)$-equivariant vector bundles.

Finally, in Section 4, in parallel with Section 3, we develop the formalism for a relative Nahm transform in the same context. We apply this formalism to transform a solution to the vortex equations over a triple, regarded as an $SU(2)$-invariant Einstein–Hermitian connection on a certain $SU(2)$-equivariant bundle over the product of the curve with the complex projective line. In general, it seems very hard to identify the equation satisfied by the Nahm transform of a vortex solution, which one would expect to be again the vortex equation on the transformed triple. We content ourselves with analysing in full detail, the case of covariantly constant triples, leaving for a future paper the analysis of the general case. As a byproduct we prove that polystability of triples may not be preserved by the Fourier–Mukai transform.

In this paper, we work over the field of complex numbers $\mathbb{C}$.

2. Fourier–Mukai and Nahm transforms on elliptic curves

2.1. Fourier–Mukai transform

Let C be an elliptic curve and let $\hat{C} = \mathrm{Pic}^0(C)$ be its dual variety. Although C and $\hat{C}$ are isomorphic, it will be convenient to keep a notational distinction between them for the sake of clarity. Over $C \times \hat{C}$ we consider the Poincaré bundle $\mathcal{P}$ and we denote by π_C and $\pi_{\hat{C}}$ the canonical projections onto the factors C and $\hat{C}$. As it is customary, $\mathcal{P}$ is normalized so that it is trivial over $\{0\} \times \hat{C}$. In [22], Mukai introduced a functor between the bounded derived categories of coherent sheaves of C and $\hat{C}$:

$$\mathcal{S} : D(C) \to D(\hat{C}).$$

This functor acts as follows

$$\mathcal{S}(E) = \mathbf{R}\pi_{\hat{C},*}(\pi_C^* E \otimes \mathcal{P}),$$

where E is an object of the derived category and $\mathbf{R}\pi_{\hat{C},*}$ denotes the derived functor of $\pi_{\hat{C},*}$.

We need some standard terminology and notation. As usual, we denote by $\mathcal{S}^i(E)$ the sheaf defined by the i-th cohomology of the complex $\mathcal{S}(E)$, that is,

$$\mathcal{S}^i(E) = \mathcal{H}^i(\mathcal{S}(E)).$$

When E is a single sheaf, $S^i(E)$ is the ordinary derived functor $R^i\pi_{\hat{C},*}(\pi_C^* E \otimes \mathcal{P})$. A sheaf E is said to be WIT_i if $\mathcal{S}^j(E) = 0$ for every $j \neq i$, and E is called IT_i if it is WIT_i and its unique transform $\mathcal{S}^i(E)$ is locally-free. Equivalently E is IT_i if the cohomology group $H^j(C_\xi, E \otimes \mathcal{P}_\xi) = 0$ vanishes for every $j \neq i$ and every $\xi \in \hat{C}$, where $C_\xi = C \times \{\xi\}$ and $\mathcal{P}_\xi$ is the restriction of $\mathcal{P}$ to C_ξ. In this case, the fibre over $\xi \in \hat{C}$ of the unique Fourier–Mukai transform $S^i(E)$ is canonically isomorphic to $H^i(C_\xi, E \otimes \mathcal{P}_\xi)$. The Fourier–Mukai transform $S^i(E)$ of a WIT_i sheaf E will be denoted as usual by $\hat{E}$. When there is no need to specify the index i we shall simply say that a sheaf is WIT or IT.

One of the most important features of the functor S is that it admits an inverse $\hat{\mathcal{S}} : D(\hat{C}) \to D(C)$. That is, there are natural isomorphisms:

$$\hat{\mathcal{S}} \circ \mathcal{S} \simeq \mathrm{Id}_{D(C)}$$
$$\mathcal{S} \circ \hat{\mathcal{S}} \simeq \mathrm{Id}_{D(\hat{C})}.$$

Moreover $\hat{\mathcal{S}}$ is explicitly given by

$$\hat{\mathcal{S}}(F) = \mathbf{R}\pi_{C,*}(\pi_{\hat{C}}^{*}(F) \otimes \mathcal{P}^{\vee}[1]),$$

where $\mathcal{P}^{\vee}$ is the dual of $\mathcal{P}$ and [1] is the shift operator.

Let us recall the following well-known fact whose proof relies on the invertibility property of the Fourier–Mukai transform (see [13], also [10,9]).

Proposition 2.1. *If E is a semistable* (*stable*) *vector bundle of non-zero degree over an elliptic curve C, then E is* IT *and the transform $\hat{E}$ is also semistable* (*stable*). *Moreover, E is* IT_0 (IT_1) *if and only if* $\deg(E) > 0$ ($\deg(E) < 0$). *Finally, if E is* IT_i *with Chern character* $\mathrm{ch}(E) = (r, d)$ *then* $\mathrm{ch}(\hat{E}) = ((-1)^i d, (-1)^{i+1} r) = (-1)^i (d, -r)$.

Remark 2.2. If we take into account that any vector bundle E on an elliptic curve decomposes uniquely into a direct sum of semistable bundles we conclude that E is IT_0 (IT_1) if and only if all of its components have positive (negative) degree.

Recall that on an elliptic curve C the moduli space $\mathcal{M}_C(r, d)$ of S-equivalence classes of semistable bundles of rank r and degree d is isomorphic to the symmetric product $S^h C$, where $h = (r, d)$ is the greatest common divisor of r and d. When $(r, d) > 1$ there are no stable bundles in $\mathcal{M}_C(r, d)$. When r and d are coprime, all the semistable bundles are stable and $\mathcal{M}_C(r, d)$ is isomorphic to C (see [1,27] for details, as well as [9,16]). The Fourier–Mukai transform is well-behaved with respect to families of stable bundles and therefore induces morphisms between moduli spaces. In the case of IT_i semistable bundles on an elliptic curve, the Fourier–Mukai transform also preserves S-equivalence. More precisely if E is an IT_i semistable bundle on C, then it is immediate to see that every stable bundle E_k in the graded object $\mathrm{Gr}(E) = \oplus_k E_k$ with respect to a Jordan–Hölder filtration is IT_i. From this follows that if E and E' are S-equivalent IT_i bundles, then the transforms $\hat{E}$ and $\hat{E}'$ remain S-equivalent. Therefore, we have:

Corollary 2.3. *Let $\mathcal{M}_C(r, d)$ be the moduli space of semistable bundles of rank r and $d \neq 0$. Then, in the* IT_i *case, the Fourier–Mukai transform induces an isomorphism between the moduli spaces*

$$\mathcal{S} : \mathcal{M}_C(r, d) \xrightarrow{\sim} \mathcal{M}_{\hat{C}}((-1)^i d, (-1)^{i+1} r).$$

Therefore, the Fourier–Mukai transform gives rise to an isomorphism between symmetric products of elliptic curves.

2.2. Nahm transform

We come now to the definition of the Nahm transform in the context of elliptic curves.

Let C be a complex elliptic curve endowed with a flat metric of unit volume. The canonical spinor bundle $S = \Lambda^{0,\bullet} T^*C$ of C as a spin^c manifold, has a natural splitting $S = S^+ \oplus S^-$ where

$$S^+ = \Lambda^{0,0}T^*C, \qquad S^- = \Lambda^{0,1}T^*C.$$

We denote the spinorial connection of S by ∇_S.

The dual elliptic curve $\hat{C}$ parametrizes the gauge equivalence classes of Hermitian flat line bundles over C. The Poincaré bundle $\mathcal{P}$ introduced in Section 2.1 is endowed with a unitary connection $\nabla_{\mathcal{P}}$, such that the restriction of $(\mathcal{P}, \nabla_{\mathcal{P}})$ to the slice C_ξ is in the equivalence class defined by $\xi \in \hat{C}$. Therefore, for every $\xi \in \hat{C}$, we have the Hermitian line bundle $\mathcal{P}_\xi \equiv \mathcal{P}_{|C_\xi} \to C$ endowed with the flat unitary connection $\overline{\nabla}_\xi = \nabla_{|\mathcal{P}_\xi}$.

Let us consider a Hermitian vector bundle $E \to C$ with a unitary connection ∇. On the vector bundle $E \otimes \mathcal{P}_\xi$, we have the connection $\nabla_\xi = \nabla \otimes 1 + 1 \otimes \overline{\nabla}_\xi$. Therefore, we have the family of coupled Dirac operators

$$D_\xi : \Omega^0(C, S^+ \otimes E \otimes \mathcal{P}_\xi) \to \Omega^0(C, S^- \otimes E \otimes \mathcal{P}_\xi).$$

It follows from the Atiyah–Singer Theorem for families that the difference bundle of the family of Dirac operators D parametrized by $\hat{C}$ is a well defined object $\mathrm{Ind}(D)$ in K-theory which is called the index of D. Moreover, if either one of $\{\mathrm{Ker}\, D_\xi\}$ or $\{\mathrm{Coker}\, D_\xi\}$ has constant rank, then $\mathrm{Ker}\, D$ and $\mathrm{Coker}\, D$ are vector bundles over $\hat{C}$ and one has that

$$\mathrm{Ind}(D) = [\mathrm{Ker}\, D] - [\mathrm{Coker}\, D] \in K(\hat{C}).$$

Definition 2.4. Let (E, ∇) be a pair formed by a Hermitian vector bundle E over C and a unitary connection ∇ on E. We say that (E, ∇) is an IT (index Theorem) pair if either $\mathrm{Coker}\, D = 0$ or $\mathrm{Ker}\, D = 0$. In the first case we say that (E, ∇) is an IT_0-pair whereas in the second we call it an IT_1-pair. The transformed bundle of an IT_i-pair is the vector bundle $\hat{E} = (-1)^i \mathrm{Ind}(D) \to \hat{C}$.

Remark 2.5. From a more formal point of view, the study of the family of Dirac operators D can be approached via the techniques developed by Bismut in his proofs of the Atiyah–Singer index Theorem for families [5]. In that framework one has to consider the fibration $\pi_{\hat{C}} : C \times \hat{C} \to \hat{C}$ as a family of spin^c manifolds, whose fibres are precisely C_ξ. The vector bundle of relative spinors is identified with $\pi_C^* S$ and we can consider then the coupled relative Dirac operator

$$D : \Omega^0(\pi_C^*(S^+ \otimes E) \otimes \mathcal{P}) \to \Omega^0(\pi_C^*(S^- \otimes E) \otimes \mathcal{P}),$$

whose restriction to C_ξ is D_ξ.

The Nahm transform from C to $\hat{C}$ is a procedure which transforms Hermitian vector bundles with unitary connections on C into Hermitian vector bundles with unitary connections on $\hat{C}$. The main idea relies on the fact that the index (minus the index) of the family D is a finite rank vector bundle whenever $\mathrm{Coker}\, D = 0$ ($\mathrm{Ker}\, D = 0$). In certain cases, this is a consequence of a vanishing Theorem of Bochner type. Before doing so, we introduce some more notation and recall the Weitzenböck formula.

We recall that the Poincaré line bundle $\mathcal{P} \to C \times \hat{C}$ is a holomorphic Hermitian line bundle and that the unitary connection $\nabla_{\mathcal{P}}$ is compatible with the holomorphic structure. It is also known that a Hermitian vector bundle $E \to C$ with a unitary connection ∇ is naturally endowed with a holomorphic structure since F^{∇} is of type $(1, 1)$ (see [12, 2.1.53]). Moreover, the spin^c Dirac operator D_ξ coincides with the Dolbeault–Dirac operator of $E \otimes \mathcal{P}_\xi$

$$D_\xi = \sqrt{2}(\overline{\partial}^*_{E\otimes\mathcal{P}_\xi} + \overline{\partial}_{E\otimes\mathcal{P}_\xi}),$$

where $\overline{\partial}_{E\otimes\mathcal{P}_\xi}$ is the Cauchy-Riemann operator of $E \otimes \mathcal{P}_\xi$. Since C is a one-dimensional complex manifold the Dolbeault–Dirac operator D_ξ is reduced to

$$D_\xi = \sqrt{2}\overline{\partial}_{E\otimes\mathcal{P}_\xi} : \Omega^0(C_\xi, E \otimes \mathcal{P}_\xi) \to \Omega^{0,1}(C_\xi, E \otimes \mathcal{P}_\xi).$$

As a consequence of the Kähler identities (see [12]), the Weitzenböck formula for the Dirac operator D_ξ can be expressed as

$$D^*_\xi D_\xi = 2\overline{\partial}^*_{E\otimes\mathcal{P}_\xi}\overline{\partial}_{E\otimes\mathcal{P}_\xi} = \nabla^*_\xi\nabla_\xi - i\Lambda F^{\nabla} \otimes \mathrm{Id}_{\mathcal{P}_\xi}, \tag{2.1}$$

where $i\Lambda F^{\nabla}$ is the Hermitian endomorphism of E obtained by contracting iF^{∇} with the Kähler form. We have the following vanishing Theorem.

Theorem 2.6. *Let (E, ∇) be a pair formed by a Hermitian vector bundle over C and a unitary connection.*

(i) *If $i\Lambda F^{\nabla}$ is non-negative and there exists $x \in C$ such that $i\Lambda F^{\nabla}(x) > 0$ then (E, ∇) is an IT_0-pair.*
(ii) *If $i\Lambda F^{\nabla}$ is a non-positive and there exists $x \in C$ such that $i\Lambda F^{\nabla}(x) < 0$ then (E, ∇) is an IT_1-pair.*

Proof. Let us suppose that $i\Lambda F^{\nabla} < 0$. If we apply the Weitzenbock formula (2.1) to a section $s \in \Gamma(C, E \otimes \mathcal{P}_\xi)$ and we integrate over C we obtain

$$\|D_\xi s\|^2 = \|\nabla_\xi s\|^2 - \int_C \langle i\Lambda F^{\nabla} s, s\rangle\, \omega \geq 0, \tag{2.2}$$

where ω is the Riemannian volume element of C. From relation (2.2) we obtain

$$D_\xi s = 0 \Longleftrightarrow \begin{cases} (a)\, \nabla_\xi s = 0 \\ (b)\, \langle i\Lambda F^{\nabla} s, s\rangle = 0. \end{cases}$$

By (a) one sees that $\langle s, s\rangle$ is constant; Therefore, if there exists $x \in C$ such that $i\Lambda F^{\nabla}(x) < 0$, then (b) implies that $s(x) = 0$ and since $\langle s, s\rangle$ is constant, one has $s = 0$ and (ii) is proved. By Serre duality, we have $H^1(C, E) \simeq H^0(C, E^{\vee})^*$, and hence the first statement follows from the second one. □

We can endow the transformed vector bundle of an IT-pair with a Hermitian metric and a unitary connection in a natural way. This follows from a rather straightforward application of the theory for families. We briefly recall the main facts of this construction following the approach of [12, Chapter 3] and [5].

Let $H^{\infty}_{\pm}$ be the space of C^{∞} sections of the vector bundle $\pi_C^*(S^{\pm}\otimes E)\otimes\mathcal{P}$ over $C\times\hat{C}$. We may regard $H^{\infty}_{\pm}$ as the space of C^{∞} sections over $\hat{C}$ of the infinite dimensional fibre bundles $\mathcal{H}^{\infty}_{\pm}$. The fibres $\mathcal{H}^{\infty}_{\pm,\xi}$ are the sets of C^{∞} sections over C_{ξ} of $\pi_C^*(S^{\pm}\otimes E)\otimes\mathcal{P}$. Since $\pi_C^*(S^{\pm}\otimes E)\otimes\mathcal{P}$ is a Hermitian vector bundle, and the fibres C_{ξ} of the projection $\pi_{\hat{C}}: C\times\hat{C}\to\hat{C}$ carry a natural volume element ω; we can define the Hermitian metric

$$\langle h_1,h_2\rangle_{\pi_{\hat{C}}}=\int_{C_{\xi}}\langle h_1,h_2\rangle\,\omega, \tag{2.3}$$

on $\mathcal{H}^{\infty}_{\pm,\xi}$, We then have the Hilbert bundles $\mathcal{H}_{\pm}$ whose fibres $\mathcal{H}_{\pm,\xi}$ are the L^2-completion of $\mathcal{H}^{\infty}_{\pm,\xi}$ with respect to this metric.

Let ∇^1 be the connection on $\pi_C^*(S^{\pm}\otimes E)\otimes\mathcal{P}$ obtained from ∇_S, ∇ and $\nabla_{\mathcal{P}}$. Now we define a connection $\tilde{\nabla}$ on $\mathcal{H}^{\infty}_{\pm}$ as follows

$$\tilde{\nabla}_D h=\nabla^1_{D^H}h,\quad \text{for every}\quad D\in\mathfrak{X}(\hat{C}),\, h\in H^{\infty}_{\pm},$$

where D^H is the natural lift of the vector field D from $\hat{C}$ to $C\times\hat{C}$. It is easy to check that $\tilde{\nabla}$ is a flat connection.

If (E,∇) is an IT_i-pair, then the regularity Theorem for elliptic operators implies that $\hat{E}$ is, according to the parity of the index i, a subbundle of $\mathcal{H}^{\infty}_{\pm}$, and hence there is a naturally induced metric on $\hat{E}$. We also have a natural unitary connection $\hat{\nabla}$ induced by the ambient connection $\tilde{\nabla}$ and the orthogonal projection P onto $\hat{E}$, that is

$$\hat{\nabla}=P\circ\tilde{\nabla}.$$

Let us recall that Hodge theory provides an explicit formula for the projector P. Indeed, if (E,∇) is IT_0 then for every $\xi\in\hat{C}$, we have

$$P_{\xi}=\mathrm{Id}-D^*_{\xi}G_{\xi}D_{\xi},$$

where G_{ξ} is the Green operator of $D_{\xi}D^*_{\xi}$. A similar formula holds in the case of an IT_1 pair.

Definition 2.7. Let (E,∇) be an IT-pair. The pair $(\hat{E},\hat{\nabla})$ is called the Nahm transform of (E,∇) and is denoted by $\mathcal{N}(E,\nabla)$.

Remark 2.8. If ∇ and ∇' are gauge equivalent unitary connections, it follows from the very definition of the Nahm transform that $\hat{\nabla}$ and $\hat{\nabla}'$ are also gauge equivalent unitary connections.

The following is an easy consequence of the flatness of $\tilde{\nabla}$.

Proposition 2.9. *Let $(\hat{E}, \hat{\nabla})$ be the Nahm transform of an* IT_i *pair. The curvature of $\hat{\nabla}$ is given by*

$$F^{\hat{\nabla}} = P \circ (\tilde{\nabla} P \wedge \tilde{\nabla} P) \circ P.$$

Moreover, we can express the curvature in terms of the Green operator as follows

$$F^{\hat{\nabla}} = P \circ (\tilde{\nabla} D^* \circ G \wedge \tilde{\nabla} D) \circ P, \quad \text{if} \quad E \text{ is } \mathrm{IT}_0.$$

A similar expression holds in the case of an IT_1 *pair.*

We study now the Nahm transform of a connection with constant central curvature. Since all the line bundles $\mathcal{P}_\xi$ are flat they are trivial as smooth bundles and we may consider the connection $\nabla_{\mathcal{P}}$ of the Poincaré line bundle as a family of connections $\overline{\nabla}_\xi$ on the trivial line bundle. In the same way if $E \to C$ is a Hermitian vector bundle with a unitary connection ∇, then we have a family of connections ∇_ξ on E and the family of Dolbeault–Dirac operators D_ξ considered above act now in the same vector bundle

$$\overline{\partial}^{\nabla_\xi} : \Omega^0(E) \to \Omega^{0,1}(E).$$

Using a flat holomorphic coordinate z on C and the flat coordinate w which it induces on $\hat{C}$, we have

$$\overline{\partial}^{\nabla_\xi} = \overline{\partial}^{\nabla} + \pi\, w d\overline{z} \otimes \mathrm{Id}_E,$$

which clearly shows that this family depends holomorphically on $w \in \hat{C}$.

The triviality of the holomorphic tangent bundle of C allows to identify $\Omega^{0,1}(C)$ with $\Omega^0(C)$ by contraction with a global anti-holomorphic vector field $\overline{V}$. Since the metric on C is flat, we can choose $\overline{V}$ such that it is a parallel vector field whose pointwise norm is equal to 1. We define the operator

$$\mathcal{D}_\xi = i_{\overline{V}} \overline{\partial}^{\nabla_\xi} : \Omega^0(E) \longrightarrow \Omega^0(E).$$

Lemma 2.10. *The curvature of ∇_ξ is related to the operator $\mathcal{D}_\xi$ by the formula*

$$i\Lambda F^{\nabla} = i\Lambda F^{\nabla_\xi} = 2[\mathcal{D}_\xi, \mathcal{D}_\xi^*].$$

Let us recall that the flat metric of C induces in a natural way a flat metric on $\hat{C}$. In the following Theorem we consider unitary connections of constant central curvature on C and $\hat{C}$ with respect to these metrics.

Theorem 2.11. *Let ∇ be a connection on E with constant central curvature with factor $\lambda \in \mathbb{R}$, that is $i\Lambda F^{\nabla} = \lambda\, \mathrm{Id}_E$, where $\lambda = 2\pi\mu(E)$ and $\mu(E)$ is the slope of E.*

1. *If* $\deg(E) > 0$ *then* (E, ∇) *is an* IT_0 *pair and* $\hat{\nabla}$ *is a connection on* $\hat{E}$ *with constant central curvature with factor* $\hat{\lambda} = -2\pi/\mu(E)$.
2. *If* $\deg(E) < 0$ *then* (E, ∇) *is an* IT_1 *pair and* $\hat{\nabla}$ *is a connection on* $\hat{E}$ *with constant central curvature with factor* $\hat{\lambda} = -2\pi/\mu(E)$.

Proof. We shall only prove the first case since the second one can be dealt with in a similar way.

It is well-known ([11,25]) that since ∇ has constant central curvature E must be polystable. The condition $\deg(E) > 0$ implies, due to Proposition 2.1, that (E, ∇) is an IT_0 pair. All the operators $\mathcal{D}_\xi$ act on $\Omega^0(E)$, Therefore, the bundle of kernels $\hat{E}$ is a finite rank subbundle of the trivial Hilbert bundle $\mathcal{H}_+ \to \hat{C}$ introduced above and $P : \mathcal{H}_+ \to \hat{E}$ is the orthogonal projection. Then, we have

$$\hat{\nabla} = P \circ \tilde{\nabla},$$

where $\tilde{\nabla}$ is the natural flat connection on $\mathcal{H}_+$. Taking into account the above identifications, the curvature of the connection $\hat{\nabla}$ of $\hat{E}$, given in Proposition 2.9, can be expressed as

$$F^{\hat{\nabla}} = P_\xi \circ (\tilde{\nabla}\mathcal{D}_\xi^* \circ G_\xi \wedge \tilde{\nabla}\mathcal{D}_\xi) \circ P_\xi, \tag{2.4}$$

where G_ξ is the Green operator of $\mathcal{D}_\xi\mathcal{D}_\xi^*$.

As we mentioned above, we can choose a flat holomorphic coordinate z on C such that the Kähler form is expressed as

$$\omega = \frac{i}{2} dz \wedge d\bar{z}.$$

Therefore, locally we may take $\bar{V} = \partial/\partial\bar{z}$. This implies that

$$\mathcal{D}_\xi = \mathcal{D}_0 + \pi w \mathrm{Id}_E.$$

It is clear now that $\tilde{\nabla}\mathcal{D}_\xi = \pi dw \otimes \mathrm{Id}_E$ and $\tilde{\nabla}\mathcal{D}_\xi^* = \pi d\bar{w} \otimes \mathrm{Id}_E$ which upon substitution in (2.4) gives

$$F^{\hat{\nabla}} = \pi^2 P_\xi \circ G_\xi \circ P_\xi \, d\bar{w} \wedge dw,$$

where we have used the fact that the identity operator commutes with the Green's operator G_ξ. We then have to prove that for every $u \in \mathrm{Ker}\mathcal{D}_\xi$ one has

$$G_\xi u = \alpha u + v,$$

where α is a constant and $v \in (\mathrm{Ker}\mathcal{D}_\xi)^\perp$. To see this suppose that

$$G_\xi u = u' + v \quad \text{for} \quad u' \in \mathrm{Ker}\,\mathcal{D}_\xi \quad \text{and} \quad v \in (\mathrm{Ker}\,\mathcal{D}_\xi)^\perp.$$

Operating by $G_\xi^{-1} = \mathcal{D}_\xi\mathcal{D}_\xi^*$ we obtain

$$u = \mathcal{D}_\xi\mathcal{D}_\xi^* u' + \mathcal{D}_\xi\mathcal{D}_\xi^* v. \tag{2.5}$$

But by hypothesis $[\mathcal{D}_0, \mathcal{D}_0^*] = \lambda/2\,\mathrm{Id}_E$, Therefore, $[\mathcal{D}_\xi, \mathcal{D}_\xi^*] = \lambda/2\,\mathrm{Id}_E$ and, since $\mathcal{D}_\xi u' = 0$, Eq. (2.5) becomes

$$u - \frac{\lambda}{2}u' = \mathcal{D}_\xi\mathcal{D}_\xi^* v.$$

Now $\mathcal{D}_\xi\mathcal{D}_\xi^* v \in (\mathrm{Ker}\,\mathcal{D}_\xi)^\perp$, since for every $u_1 \in \mathrm{Ker}\,\mathcal{D}_\xi$,

$$(\mathcal{D}_\xi\mathcal{D}_\xi^* v, u_1) = \left(\frac{\lambda}{2}v + \mathcal{D}_\xi^*\mathcal{D}_\xi v, u_1\right) \quad = \left(\frac{\lambda}{2}v, u_1\right) + (\mathcal{D}_\xi v, \mathcal{D}_\xi u_1) \quad = 0.$$

Thus $u - \lambda/2\,u' \in \mathrm{Ker}\,\mathcal{D}_\xi \cap (\mathrm{Ker}\mathcal{D}_\xi)^\perp = \{0\}$. Hence $u' = 2\lambda^{-1}u$, concluding that

$$F^{\hat\nabla} = -2\pi^2\lambda^{-1}dw \wedge d\overline{w} = -\frac{(2\pi)^2}{i\lambda}\frac{i}{2}dw \wedge d\overline{w} = -\frac{(2\pi)^2}{i\lambda}\hat\omega \otimes \mathrm{Id}_{\hat E},$$

where $\hat\omega$ is the Kähler form of $\hat C$. Therefore $i\Lambda F^{\hat\nabla} = -2\pi/\mu(E)\,\mathrm{Id}_{\hat E}$ as required. □

2.3. Compatibility between the Fourier–Mukai and Nahm transforms, functoriality and invertibility

Let $E \to C$ be a Hermitian vector bundle endowed with a unitary connection ∇. As we have seen the spin^c Dirac operator D_ξ is identified with the Dolbeault–Dirac operator of $E \otimes \mathcal{P}_\xi$. Hodge theory and the Dolbeault isomorphism give that

$$\mathrm{Ker}\,D_\xi \simeq H^0(C_\xi, E \otimes \mathcal{P}_\xi) \tag{2.6}$$

$$\mathrm{Coker}\,D_\xi \simeq H^1(C_\xi, E \otimes \mathcal{P}_\xi). \tag{2.7}$$

If we suppose that E is IT_i with respect to the Fourier–Mukai transform $\mathcal{S}$, then the isomorphisms (2.6) and (2.7) mean that (E, ∇) is an IT_i-pair with respect to the Nahm transform. By [2, Theorem 2] or [12, Theorem 3.2.8], we have a natural C^∞ vector bundle isomorphism induced by Hodge theory

$$\phi_E : \hat E \xrightarrow{\sim} \mathcal{S}^i(E).$$

Moreover, we have the following.

Theorem 2.12. *Let E_1, E_2 be Hermitian vector bundles over C endowed with unitary connections ∇_1, ∇_2 such that (E_1, ∇_1), (E_2, ∇_2) are IT_i-pairs with respect to the Nahm transform. Then, we have*

1. *The connections* $\hat{\nabla}_1$, $\hat{\nabla}_2$ *are compatible with the holomorphic structures of* $\mathcal{S}^i(E_1)$, $\mathcal{S}^i(E_2)$, *respectively.*
2. *For every holomorphic morphism* $\Phi : E_2 \to E_1$ *we have an induced holomorphic morphism* $\mathcal{N}(\Phi) : \hat{E}_2 \to \hat{E}_1$ *and a commutative diagram*

$$\begin{array}{ccc} \widehat{E}_2 & \xrightarrow{\phi_{E_2}} & \mathcal{S}(E_2) \\ {\scriptstyle \mathcal{N}(\Phi)}\downarrow & & \downarrow{\scriptstyle \mathcal{S}(\Phi)} \\ \widehat{E}_1 & \xrightarrow{\phi_{E_1}} & \mathcal{S}(E_1) \end{array} \tag{2.8}$$

Proof. The Poincaré bundle $\mathcal{P} \to C \times \hat{C}$ is a holomorphic line bundle and the connection $\nabla_{\mathcal{P}}$ is compatible with the holomorphic structure. This implies that the families of Dirac operators D_ξ, D_ξ^* vary holomorphically with $\xi \in \hat{C}$. The first statement follows now by a standard argument concerning holomorphic families, see [12, Theorem 3.2.8].

Since Φ is holomorphic, the second statement follows immediately in the IT_0 case, because the fibers of the Nahm transforms are given by the kernels of the Cauchy-Riemann operators. In the IT_1 case the fibers $\hat{E}_{2,\xi}$, $\hat{E}_{1,\xi}$ of the Nahm transform at $\xi \in \hat{C}$ are given by the cokernels of the Dirac operators $D_{2,\xi}$, $D_{1,\xi}$ and we have

$$\mathrm{Coker}\, D_{2,\xi} = \mathrm{Ker}\, D_{2,\xi}^* = \mathrm{Ker}\, \overline{\partial}^{\nabla_{2,\xi}^*}$$

$$\mathrm{Coker}\, D_{1,\xi} = \mathrm{Ker}\, D_{1,\xi}^* = \mathrm{Ker}\, \overline{\partial}^{\nabla_{1,\xi}^*}$$

Now Φ induces a morphism from $\mathrm{Ker}\, \overline{\partial}^{\nabla_{2,\xi}^*}$ to $\Omega^{0,1}(C, E_1 \otimes \mathcal{P}_\xi)$ and composing it with the orthogonal projection onto $\mathrm{Ker}\, \overline{\partial}^{\nabla_{1,\xi}^*}$ we get a morphism

$$\mathcal{N}(\Phi)_\xi : \mathrm{Ker}\, \overline{\partial}^{\nabla_{2,\xi}^*} \to \mathrm{Ker}\, \overline{\partial}^{\nabla_{1,\xi}^*}$$

which by Hodge theory is the unique one that renders commutative the following diagram

$$\begin{array}{ccc} \widehat{E}_{2,\xi} & \xrightarrow{\phi_{E_2,\xi}} & \mathcal{S}(E_2)_\xi = H^1(C, E_{2,\xi}) \\ {\scriptstyle \mathcal{N}(\Phi)_\xi}\downarrow & & \downarrow{\scriptstyle \mathcal{S}(\Phi)_\xi} \\ \widehat{E}_{1,\xi} & \xrightarrow{\phi_{E_1,\xi}} & \mathcal{S}(E_1)_\xi = H^1(C, E_{1,\xi}) \end{array}$$

Since $\mathcal{S}(\Phi)$ is a vector bundle morphism and ϕ_{E_2}, ϕ_{E_1} are C^∞ vector bundle isomorphisms, we conclude that $\mathcal{N}(\Phi)$ is also a C^∞ vector bundle morphism and we have the commutative diagram (2.8). Moreover, $\mathcal{S}(\Phi)$ is a holomorphic morphism and by the first part of the Theorem, we have the compatibility between the connections $\hat{\nabla}_2$, $\hat{\nabla}_1$ and the holomorphic structures of $\mathcal{S}(E_2)$, $\mathcal{S}(E_1)$, respectively. These facts imply that $\mathcal{N}(\Phi)$ is a holomorphic morphism. □

If h is an Hermitian metric on a C^∞ vector bundle E then $\mathcal{A}(E, h)$ will denote the space of unitary connections which are compatible with h. On the other hand we will denote by

$\mathcal{C}(E)$ the set of holomorphic structures on E. It is well known that there is an identification

$$\mathcal{A}(E,h) \xrightarrow{\sim} \mathcal{C}(E)$$

which associates to (E, ∇) the holomorphic vector bundle $\mathcal{E} = (E, \overline{\partial}^{\nabla})$, the inverse correspondence being given by the map which associates to every holomorphic bundle $\mathcal{E} = (E, \overline{\partial}^{\nabla})$ the unique connection ∇ compatible with the complex structure and the Hermitian metric. We can rephrase the preceding Theorem by saying that the Nahm transform and the Fourier–Mukai transform are compatible with this identification. That is to say, the following diagram is commutative

$$\begin{tikzcd} \mathcal{A}(E,h) \arrow[r, "\sim"] \arrow[d, "\mathcal{N}"'] & \mathcal{C}(E) \arrow[d, "\mathcal{S}"] \\ \mathcal{A}(E,h) \arrow[r, "\sim"] & \mathcal{C}(E). \end{tikzcd}$$

The curve C and its dual elliptic curve $\hat{C}$ are in a symmetrical dual relation with one another (see [12, Section 3.3.2]). That is, C parametrizes the flat Hermitian line bundles over $\hat{C}$, therefore $\hat{\hat{C}} \simeq C$. Moreover, the restriction of the dual of the Poincaré line bundle $\mathcal{P}^{\vee}$ to the slice $\hat{C}_x$ endowed with the restriction of the connection $\nabla_{\mathcal{P}^{\vee}}$ is isomorphic, as a Hermitian line bundle with connection, to the flat Hermitian bundle corresponding to x. We can hence apply the Nahm construction in order to transform Hermitian vector bundles with connection over $\hat{C}$ into Hermitian vector bundles with connection over C.

Let ∇ be a connection with constant central curvature different from zero on a bundle E over C, and let $\mathcal{E} = (E, \overline{\partial}^{\nabla})$ be the corresponding holomorphic vector bundle; then $\deg(\mathcal{E}) \neq 0$. The isomorphisms (2.6) and (2.7) imply that $\mathcal{E}$ is IT_i with respect to the Fourier–Mukai transform. Let $\hat{\mathcal{E}} = \mathcal{S}^i(E)$ be its unique transform. It is well known, see [22], that $\hat{\mathcal{E}}$ is IT_{1-i} and that there is an isomorphism of holomorphic vector bundles

$$\hat{\hat{\mathcal{E}}} = \hat{\mathcal{S}}^{1-i}(\mathcal{S}^i(E)) \simeq \mathcal{E}. \tag{2.9}$$

By Theorem 2.11 $\hat{\nabla}$ is a constant central curvature connection on $\hat{E}$, and hence we can apply to it the Nahm transform to obtain $(\hat{\hat{E}}, \hat{\hat{\nabla}})$. By (2.9) we have an isomorphism

$$\hat{\hat{E}} \simeq E.$$

Moreover, Theorem 2.12 implies that $\hat{\hat{\nabla}}$ is compatible with the holomorphic structure of $\mathcal{E}$, and therefore by the results of Donaldson [11], which in particular extend the theorem of Narasimhan and Seshadri [25] to genus one, we have the following.

Theorem 2.13. *If ∇ is a connection with constant central curvature different from zero on E then $\hat{\nabla}$ is a connection with constant central curvature on the bundle $\hat{E}$, and there is a natural isomorphism*

$$(\hat{\hat{E}}, \hat{\hat{\nabla}}) \simeq (E, \nabla).$$

Let $\mathcal{A}_c(E, h) \subset \mathcal{A}(E, h)$ be the subspace of constant central curvature connections and let $\mathcal{C}_{ps}(E) \subset \mathcal{C}(E)$ be the subspace of polystable holomorphic structures on the C^∞ bundle E. We have the Donaldson–Narasimhan–Seshadri correspondence (the curve version of the Hitchin–Kobayashi correspondence)

$$\mathcal{A}_c(E, h) \xrightarrow{D} \mathcal{C}_{ps}(E).$$

The content of the preceding Theorem can be summarized by saying that the Nahm transform and the Fourier–Mukai transform are compatible with the Donaldson–Narasimhan–Seshadri correspondence. That is, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{A}_c(E,h) & \xrightarrow{D} & \mathcal{C}_{ps}(E) \\ {\scriptstyle\mathcal{N}}\downarrow & & \downarrow{\scriptstyle\mathcal{S}} \\ \mathcal{A}_c(\widehat{E},\hat{h}) & \xrightarrow{D} & \mathcal{C}_{ps}(\widehat{E}) \,. \end{array}$$

These correspondences descend to the quotients by the corresponding gauge groups, giving a commutative diagram of correspondences between the associated moduli spaces. First, the Donaldson–Narasimhan–Seshadri correspondence is well known to descend to moduli spaces, see [20, Chapter VII]. The descent for the Nahm transform follows from Remark 2.8 and for the Fourier–Mukai transform is a consequence of its functoriality.

3. Fourier–Mukai transforms for holomorphic triples

3.1. Holomorphic triples

A holomorphic triple over a smooth connected curve C is by definition a triple $T = (E_1, E_2, \Phi)$ where E_i, $i = 1, 2$ are holomorphic vector bundles and $\Phi \in \mathrm{Hom}_C(E_2, E_1)$. Let n_i and d_i be the rank and degree of E_i for $i = 1, 2$. We say that the triple T is of type (n_1, n_2, d_1, d_2). There is a notion of stability for triples which depends on a real parameter α (see [7] for details). The α-degree of T is defined by

$$\deg_\alpha(T) = \deg(E_1 \oplus E_2) + n_2\alpha$$

and the α-slope is by definition

$$\mu_\alpha(T) = \frac{\deg_\alpha(T)}{n_1 + n_2}.$$

The stability condition is defined in a similar way as the slope stability for vector bundles, precisely: $T = (E_1, E_2, \Phi)$ is α-stable (respectively α-semistable) if for every non-trivial subtriple $T' = (E'_1, E'_2, \Phi')$ we have

$$\mu_\alpha(T') < \mu_\alpha(T) \qquad \text{(respectively } \leq\text{)}.$$

Here a subtriple means a triple $T' = (E_1', E_2', \Phi')$ and injective homomorphisms γ_1, γ_2 of sheaves such that the following diagram commutes

$$\begin{tikzcd} E_2' \arrow[r, "\Phi'"] \arrow[d, "\gamma_2"'] & E_1' \arrow[d, "\gamma_1"] \\ E_2 \arrow[r, "\Phi"] & E_1. \end{tikzcd}$$

Most of the properties which are valid for stable bundles carry along to stable triples. We denote the moduli space of S-equivalence classes of α-semistable triples of type (n_1, n_2, d_1, d_2) by $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ or simply by $\mathcal{N}_\alpha$ if there is no need to specify the topological invariants. $\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ denotes the moduli space of α-stable triples.

An important feature is that the stability condition gives bounds on the range of the parameter α. More precisely, if $n_1 \neq n_2$ and $T = (E_1, E_2, \Phi)$ is α-stable of type (n_1, n_2, d_1, d_2) then necessarily

$$0 \leq \alpha_m \leq \alpha \leq \alpha_M$$

where $\alpha_m = \mu_1 - \mu_2$ and $\alpha_M = (1 + (n_1 + n_2)/(|n_1 - n_2|))(\mu_1 - \mu_2)$ (see [7]). In the case $n_1 = n_2$, α ranges in $[\alpha_m, \infty)$; we will write in this case, $\alpha_M = \infty$. The interval (α_m, α_M) is divided into a finite number of subintervals determined by values of the parameter for which strict semistability may occur. The stability criteria for two values of α lying between two consecutive critical values are equivalent (and therefore the corresponding moduli spaces are isomorphic). As in [8] we shall denote by α_L the largest critical value, in particular when $\alpha_L < \alpha < \alpha_M$ all the moduli spaces $\mathcal{N}_\alpha$ are isomorphic.

We use freely the terminology and results of [7]. Corollary 3.6, Proposition 3.17, Corollaries 3.19 and 3.20 and Lemma 4.6 of [7] are particularly useful for the understanding of this paper.

Now we recall how holomorphic triples on an elliptic curve C are related to $SU(2)$-equivariant bundles on the elliptic surface $C \times \mathbb{P}^1$. In what follows we shall only deal with $SU(2)$-equivariant bundles E which admit a C^∞ $SU(2)$-equivariant decomposition of the type

$$E = p^* E_1 \oplus (p^* E_2 \otimes q^* H^{\otimes 2}), \tag{3.1}$$

where p, q are the canonical projections of $C \times \mathbb{P}^1$ onto its factors and H is the C^∞ line bundle over $\mathbb{P}^1$ with first Chern number equal to 1.

In the following, if not otherwise stated, an $SU(2)$-equivariant bundle will always mean an holomorphic bundle over $C \times \mathbb{P}^1$, $SU(2)$-equivariant, of type given in (3.1).

We shall need the following formulation of Proposition 2.3 in [7].

Proposition 3.1. *Let C be a smooth connected curve, then*

(i) *There is a one-to-one correspondence between $SU(2)$-equivariant holomorphic vector bundles E of type* (3.1) *and holomorphic extensions over $C \times \mathbb{P}^1$ of the form*

$$0 \to p^* E_1 \to E \to p^* E_2 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

where E_1, E_2 are holomorphic vector bundles on C. Here $\mathcal{O}_{\mathbb{P}^1}(2)$ is the unique line bundle of degree 2 over $\mathbb{P}^1$.

(ii) *There is a* (*non-unique*) *functorial correspondence between such extensions and elements of* $\mathrm{Hom}_C(E_2, E_1)$ *and it is given by a functorial isomorphism*

$$\sigma_C : \mathrm{Ext}^1_{C\times\mathbb{P}^1}(p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*E_1) \simeq \mathrm{Hom}_C(E_2, E_1)$$

induced by the choice of a trace isomorphism $\mathrm{tr} : H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \tilde{\to} \mathbb{C}$.

Proof. A proof of (i) and (ii) can be found in [14, Proposition 3.9] and [7, Proposition 2.3]. We recall that there is a natural isomorphism (see for instance [15])

$$\mathrm{Ext}^1_{C\times\mathbb{P}^1}(p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*E_1) \tilde{\to} \mathrm{Hom}_{D(C\times\mathbb{P}^1)}(p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*E_1[1]). \tag{3.2}$$

Now we have

$$\begin{aligned}
&\mathrm{Hom}_{D(C\times\mathbb{P}^1)}(p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*E_1[1])\\
&\tilde{\to}\mathrm{Hom}_{D(C\times\mathbb{P}^1)}(p^*E_2, p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)[1])\\
&\tilde{\to}\mathrm{Hom}_{D(C)}(E_2, \mathbf{R}p_*(p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2))[1])\\
&\tilde{\to}\mathrm{Hom}_{D(C)}(E_2, E_1 \otimes \mathbf{R}p_*(q^*\mathcal{O}_{\mathbb{P}^1}(-2))[1])\\
&\tilde{\to}\mathrm{Hom}_{D(C)}(E_2, E_1 \otimes_{\mathbb{C}} \mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))[1])\\
&\tilde{\to}\mathrm{Hom}_{D(C)}(E_2, E_1 \otimes_{\mathbb{C}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)))
\end{aligned}$$

where the second isomorphism is adjunction between direct and inverse images, the third is the projection formula, the fourth is base-change in the derived category and the last is due to the fact that since $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, then $\mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \tilde{\to} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))[-1]$ in the derived category. Composition with a trace map $\mathrm{tr} : H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \tilde{\to} \mathbb{C}$ gives the isomorphism

$$\sigma_C : \mathrm{Ext}^1_{C\times\mathbb{P}^1}(p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*E_1) \simeq \mathrm{Hom}_C(E_2, E_1)$$

of the statement. □

Remark 3.2. We can describe quite easily in an explicit form the inverse isomorphism σ_C^{-1}. The inverse of the trace $\mathrm{tr}^{-1} : \mathbb{C} \tilde{\to} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$ defines an element of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$ and via the isomorphism

$$\begin{aligned}
\mathrm{Hom}_{D(\mathbb{P}^1)}(\mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1}[1]) &\tilde{\to} \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1}) \tilde{\to} \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2))\\
&\tilde{\to} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))
\end{aligned}$$

induces a morphism $\mathrm{tr}^{-1} : \mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1}[1]$ in the derived category. Thus, given a morphism $\Phi : E_2 \to E_1$, one finds that $\sigma_C^{-1}(\Phi)$ is the element of $\mathrm{Ext}^1_{C\times\mathbb{P}^1}(p^*E_2 \otimes$

$q^*\mathcal{O}_{\mathbb{P}^1}(2),\ p^*E_1)$ corresponding to the morphism

$$p^*(\Phi)\otimes q^*(\mathrm{tr}^{-1}) : p^*E_2\otimes q^*\mathcal{O}_{\mathbb{P}^1}(2)\to\ p^*E_1[1]$$

by the isomorphism (3.2).

Remark 3.3. Two triples (E_1, E_2, Φ) and $(E_1, E_2, \lambda\Phi)$ $(\lambda \neq 0)$ define different extensions though the same holomorphic bundle. However, they define different $SU(2)$-equivariant holomorphic vector bundles (see [13,7]), because extensions correspond to $SU(2)$-equivariant holomorphic vector bundles and not merely to holomorphic vector bundles.

The correspondence in Proposition 3.1 also preserves stability. Let ω_α be the Kähler class over $X\times\mathbb{P}^1$ defined by $\omega_\alpha = \alpha/2\, p^*\omega_C + q^*\omega_{\mathbb{P}^1}$, with $\alpha\in\mathbb{R}^+$. The following result is proved in [7].

Theorem 3.4. *Let $T = (E_1, E_2, \Phi)$ be an holomorphic triple over a smooth connected curve C and let E be the holomorphic $SU(2)$-equivariant bundle $C\times\mathbb{P}^1$ defined in Proposition 3.1. Then, if E_1 and E_2 are not isomorphic, T is α-stable if and only if E is slope-stable with respect to the Kähler form ω_α. In the case $E_1\simeq E_2$ then T is α-stable if and only if $\Phi\neq 0$, $E_1\simeq E_2$ is stable and E decomposes as a direct sum*

$$E\simeq (p^*E_1\otimes q^*\mathcal{O}_{\mathbb{P}^1}(1))\oplus(p^*E_2\otimes q^*\mathcal{O}_{\mathbb{P}^1}(1)).$$

Remark 3.5. The proof of statement (ii) in Proposition 3.1 that we have just given above shows that the correspondence between triples and equivariant bundles also extends to families. Indeed, families of (stable) triples correspond functorially to families of $SU(2)$-equivariant (stable) bundles. This implies that the moduli space $\mathcal{N}_\alpha$ of α-stable triples (of a given topological type) over an elliptic curve corresponds, via the canonical isomorphism of Proposition 3.1, to a component of the moduli space $\mathcal{M}_\alpha^{SU(2)}$ of $SU(2)$-equivariant bundles (defined by the lift of the $SU(2)$ action determined by (3.1)) stable with respect to the Kähler form ω_α. Therefore, we have a canonical identification

$$\mathcal{N}_\alpha \xrightarrow{\sim} \mathcal{M}_\alpha^{SU(2)}.$$

3.2. Fourier–Mukai transforms for triples

We begin by briefly recalling the main properties of relative Fourier–Mukai transform in the case of a trivial elliptic fibration over the projective line.

The corresponding functor is then

$$\begin{aligned}&\mathcal{S}_{\mathbb{P}^1} : D(C\times\mathbb{P}^1)\to\ D(\hat{C}\times\mathbb{P}^1)\\&\mathcal{S}_{\mathbb{P}^1}(-) = \mathbf{R}\pi_{\hat{C}\times\mathbb{P}^1,*}(\pi^*_{C\times\mathbb{P}^1}(-)\otimes\pi^*_{C\times\hat{C}}(\mathcal{P}))\end{aligned}\tag{3.3}$$

where $\pi_{C\times\mathbb{P}^1}, \pi_{\hat{C}\times\mathbb{P}^1}$ and $\pi_{C\times\hat{C}}$ are the canonical projections of $C\times\hat{C}\times\mathbb{P}^1$ onto its factors. As in Section 2.1, this functor is invertible (see for instance [16,23]).

We also know that the relative Fourier–Mukai transform is compatible with base-change in the derived category [16]. In particular, for vector bundles E in $D(C)$ and F in $D(\mathbb{P}^1)$ the base change isomorphism can be described as follows. Let us denote by $\hat{p}$, $\hat{q}$ the projections of $\hat{C} \times \mathbb{P}^1$ onto its factors. Then

$$\begin{aligned}\mathcal{S}_{\mathbb{P}^1}(p^*E \otimes q^*F) &\tilde{\to} \mathbf{R}\pi_{\hat{C}\times\mathbb{P}^1,*}(\pi^*_{C\times\mathbb{P}^1}(p^*E \otimes q^*F) \otimes \pi^*_{C\times\hat{C}}(\mathcal{P}))\\ &\tilde{\to} \mathbf{R}\pi_{\hat{C}\times\mathbb{P}^1,*}(\pi^*_{\hat{C}\times\mathbb{P}^1}(\hat{q}^*F) \otimes \pi^*_{C\times\hat{C}}(\pi^*_C E \otimes \mathcal{P}))\\ &\tilde{\to} \mathbf{R}\pi_{\hat{C}\times\mathbb{P}^1,*}(\pi^*_{C\times\hat{C}}(\pi^*_C E \otimes \mathcal{P})) \otimes \hat{q}^*(F)\\ &\tilde{\to} \hat{p}^*(\mathbf{R}\pi_{\hat{C},*}(\pi^*_C(E) \otimes \mathcal{P})) \otimes \hat{q}^*(F) = \hat{p}^*(\mathcal{S}(E)) \otimes \hat{q}^*(F)\end{aligned} \tag{3.4}$$

where the second isomorphism is due to $q \circ \pi_{C\times\mathbb{P}^1} = \hat{q} \circ \pi_{\hat{C}\times\mathbb{P}^1}$ and $p \circ \pi_{C\times\mathbb{P}^1} = \pi_C \circ \pi_{\hat{C}\times\hat{C}}$, the third is the projection formula and the forth is base change in the derived category. We also see that given morphisms $\Phi : E_2 \to E_1$ of vector bundles on C and $\gamma : F_2 \to F_1$ of vector bundles on $\mathbb{P}^1$, then the morphism $\mathcal{S}_{\mathbb{P}^1}(p^*\Phi \otimes q^*\gamma)$ is identified with $\hat{p}^*(\mathcal{S}(\Phi)) \otimes \hat{q}^*\gamma$, that is, the following diagram is commutative

$$\begin{array}{ccc}\mathcal{S}_{\mathbb{P}^1}(p^*E_2 \otimes q^*F_2) & \xrightarrow{\mathcal{S}_{\mathbb{P}^1}(p^*\Phi\otimes q^*\gamma)} & \mathcal{S}_{\mathbb{P}^1}(p^*E_1 \otimes q^*F_1)\\ \wr\downarrow & & \downarrow\wr\\ \hat{p}^*(\mathcal{S}(E_2)) \otimes \hat{q}^*(F_2) & \xrightarrow{\hat{p}^*(\mathcal{S}(\Phi))\otimes\hat{q}^*\gamma} & \hat{p}^*(\mathcal{S}(E_1)) \otimes \hat{q}^*(F_1)\end{array} \tag{3.5}$$

where the vertical isomorphisms are the base change isomorphisms (3.4) we have just considered.

We shall give two natural definitions of the Fourier–Mukai transform of a triple and show that they are equivalent under the isomorphism given in (ii) of Proposition 3.1. First we must ensure that the transform of a triple is again a triple.

Definition 3.6. The triple $T = (E_1, E_2, \Phi)$ is IT_i if both bundles E_1, E_2 are IT_i with the same index i.

Definition 3.7 (I). Let $T = (E_1, E_2, \Phi)$ be an IT_i triple. The Fourier–Mukai transform of T is defined as the triple $\hat{T} = (\mathcal{S}^i(E_1), \mathcal{S}^i(E_2), \mathcal{S}^i(\Phi))$. We shall write $\hat{T} = (\hat{E}_1, \hat{E}_2, \hat{\Phi})$ for the transformed triple.

Since a triple T corresponds exactly to an $SU(2)$-equivariant bundle E on $C \times \mathbb{P}^1$, this suggests another definition of the Fourier–Mukai transform of an IT_i triple as the triple associated to the transform of the bundle E with respect to the relative transform $\mathcal{S}_{\mathbb{P}^1}$. This observation leads us in a natural way to consider a relative version of the Nahm transform, an argument that will be pursued in the next section. Note that in order that the relative transform of the bundle E consists of a single sheaf, we should ensure that E is IT_i. This is achieved by the following Proposition whose proof is a straightforward consequence of the base change property of the Fourier–Mukai transform.

Proposition 3.8. *If E_1 and E_2 are IT_i-bundles with respect to $\mathcal{S}$ (with the same index i), then E is IT_i with respect to $\mathcal{S}_{\mathbb{P}^1}$ and its transform $\hat{E}$ sits in an exact sequence of the type*

$$0 \to \hat{p}^*\hat{E}_1 \to \hat{E} \to \hat{p}^*\hat{E}_2 \otimes \hat{q}^*\mathcal{O}_{\mathbb{P}^1}(2) \to 0.$$

Therefore, $\hat{E}$ is an $SU(2)$-equivariant bundle on $\hat{C} \times \mathbb{P}^1$.

Now we can define.

Definition 3.9 (II). We define the Fourier–Mukai transform of a IT_i triple $T = (E_1, E_2, \Phi)$ as the triple associated to the transform $\hat{E} = \mathcal{S}^i_{\mathbb{P}^1}(E)$ of the associated $SU(2)$-equivariant and IT_i bundle E on $C \times \mathbb{P}^1$.

It remains to check that Definitions 3.7 and 3.9 are compatible.

Proposition 3.10. *Let T be an* IT *triple and let E be the corresponding invariant bundle on $C \times \mathbb{P}^1$, then the (absolute) Fourier–Mukai transform $\hat{T}$ in Definition 3.7 corresponds to the triple given by the transform $\hat{E}$ of Definition 3.9 under the isomorphism given in Proposition 3.1. In other words, we have the following commutative diagram*

$$\begin{tikzcd}
\mathrm{Ext}^1_{C\times\mathbb{P}^1}(p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*E_1)) \arrow[r, "\mathcal{S}_{\mathbb{P}^1}"] \arrow[d, "\sigma_C\ \wr"'] & \mathrm{Ext}^1_{C\times\mathbb{P}^1}(\hat{p}^*\widehat{E}_2 \otimes \hat{q}^*\mathcal{O}_{\mathbb{P}^1}(2), \hat{p}^*\widehat{E}_1)) \arrow[d, "\wr\ \sigma_{\hat{C}}"] \\
\mathrm{Ext}^0_C(E_2, E_1) \arrow[r, "\mathcal{S}"] & \mathrm{Ext}^0_C(\widehat{E}_2, \widehat{E}_1)
\end{tikzcd}$$

where the vertical rows are the isomorphisms introduced in Proposition 3.1 and the horizontal isomorphisms are induced from the relative and absolute Fourier–Mukai transforms.

Proof. Given a morphism $\Phi : E_2 \to E_1$, we know by Remark 3.2, that $\sigma_C^{-1}(\Phi)$ is the element of $\mathrm{Ext}^1_{C\times\mathbb{P}^1}(p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*E_1)$ corresponding by (3.2) to the morphism

$$p^*(\Phi) \otimes q^*(\mathrm{tr}^{-1}) : p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2) \to p^*E_1[1].$$

Now, by (3.5), $\mathcal{S}_{\mathbb{P}^1}(\sigma_C^{-1}(\Phi))$ is the element of $\mathrm{Ext}^1_{\hat{C}\times\mathbb{P}^1}(\hat{p}^*\mathcal{S}(E_2) \otimes \hat{q}^*\mathcal{O}_{\mathbb{P}^1}(2), \hat{p}^*\mathcal{S}(E_1))$ corresponding by (3.2) to the morphism

$$\hat{p}^*(\mathcal{S}(\Phi)) \otimes \hat{q}^*(\mathrm{tr}^{-1}) : \hat{p}^*\mathcal{S}(E_2) \otimes \hat{q}^*\mathcal{O}_{\mathbb{P}^1}(2) \to \hat{p}^*\mathcal{S}(E_1)[1]$$

which, again by Remark 3.2, corresponds to $\sigma_{\hat{C}}^{-1}(\mathcal{S}(\Phi))$. $\square$

Remark 3.11. In order to ensure that the Fourier–Mukai transform gives rise to morphisms between moduli spaces of triples one should check that the transform preserves families of (IT) triples. This can be checked directly as in the usual case of families of sheaves, alternatively one can use Remark 3.5 and note that the Fourier–Mukai transform is well-behaved with respect to families and therefore induces morphisms between the moduli spaces of $SU(2)$-equivariant sheaves.

3.3. Preservation of stability for small α

Let $\mathcal{N}^s_{\alpha_m^+}(n_1, n_2, d_1, d_2)$ be the moduli space of α_m^+-stable triples with $\alpha_m^+ = \alpha_m + \epsilon$ such that $\epsilon > 0$ and $(\alpha_m, \alpha_m^+]$ does not contain any critical value. (We assume that $d_1/n_1 \geq d_2/n_2$, since this is a necessary condition for the moduli space not to be empty.) One has the following (Proposition 3.23 in [7]).

Proposition 3.12. *If a triple $T = (E_1, E_2, \Phi)$ is α_m^+-stable, E_1 and E_2 are semistable. Conversely, if E_1 and E_2 are stable and $\Phi \neq 0$ then $T = (E_1, E_2, \Phi)$ is α_m^+-stable.*

Proposition 3.13. *If $(n_1, d_1) = 1$, $(n_2, d_2) = 1$ and $d_1/n_1 > d_2/n_2$, the moduli space of stable triples $\mathcal{N}^s_{\alpha_m^+}$ is isomorphic to a $\mathbb{P}^N$-fibration over $\mathcal{M}_C(n_1, d_1) \times \mathcal{M}_C(n_2, d_2)$, where $N = n_2 d_1 - n_1 d_2 - 1$.*

Proof. By Proposition 3.12, $\mathcal{N}^s_{\alpha_m^+}$ is the projectivization of a Picard sheaf on $\mathcal{M}_C(n_1, d_1) \times \mathcal{M}_C(n_2, d_2)$ (Corollary 6.2 in [8]), which in this case is a vector bundle with fibre $H^0(C, E_2^* \otimes E_1)$ over (E_1, E_2), since $H^1(C, E_2^\vee \otimes E_1) \simeq H^0(C, E_1^\vee \otimes E_2)^* = 0$. □

Given an IT_i triple $T = (E_1, E_2, \Phi)$ with transform $\hat{T} = (\hat{E}_1, \hat{E}_2, \hat{\Phi})$ we denote by $\hat{\alpha}_m$ the minimum value of the stability parameter $\hat{\alpha}$ with the type $(\hat{n}_1, \hat{n}_2, \hat{d}_1, \hat{d}_2)$ defined by $\hat{T}$. As above, $\hat{\alpha}_m^+$ is any real number such that the interval $(\hat{\alpha}_m, \hat{\alpha}_m^+]$ does not contain critical values.

Theorem 3.14. *Let $T = (E_1, E_2, \Phi)$ be a α_m^+-stable triple such that $(n_1, d_1) = 1$, $(n_2, d_2) = 1$ and $d_1 d_2 > 0$ (this forces $\Phi \neq 0$). Then the Fourier–Mukai transform $\hat{T} = (\hat{E}_1, \hat{E}_2, \hat{\Phi})$ is $\hat{\alpha}_m^+$-stable. The result also holds in the converse direction with the obvious modifications on the hypotheses.*

Proof. By Proposition 3.12, we have that E_1 and E_2 are both semistable. Moreover, E_1 and E_2 are stable due to the conditions on the rank and degree. Thus in the triple $\hat{T} = (\hat{E}_1, \hat{E}_2, \hat{\Phi})$ both bundles are stable. By Proposition 3.12 again we conclude that the triple $\hat{T}$ is also α_m^+-stable. The proof of the converse is identical. □

Corollary 3.15. *Keeping the conditions stated in the previous Theorem and assuming additionally that $d_1/n_1 > d_2/n_2$, then the Fourier–Mukai transform induces an isomorphism*

$$\mathcal{S} : \mathcal{N}^s_{\alpha_m^+} \xrightarrow{\sim} \mathcal{N}^s_{\hat{\alpha}_m^+}.$$

In other words, the Fourier–Mukai transform induces an isomorphism between the $\mathbb{P}^N$-fibrations described in Proposition 3.13.

3.4. Preservation of stability for large α

Recall that α_L is the largest critical value in the interval (α_m, α_M). If $\alpha_L < \alpha < \alpha_M$ the stability condition does not vary in this range, and we can then denote by $\mathcal{N}^s_{\alpha_M^-}(n_1, n_2, d_1, d_2)$ the moduli space of α-stable triples for any value $\alpha \in (\alpha_L, \alpha_M)$.

The relationship between the stability of the triple and that of the involved bundles is given by the following Proposition ([8, Propositions 7.5 and 7.6]).

Proposition 3.16. *Let $T = (E_1, E_2, \Phi)$ be an α-semistable triple for some $\alpha_L < \alpha < \alpha_M$, and let us suppose that $n_1 > n_2$. Then T defines an extension of the form*

$$0 \to E_2 \stackrel{\Phi}{\to} E_1 \to F \to 0 \tag{3.6}$$

with F locally free, and E_2 and F are semistable. Conversely, let $T = (E_1, E_2, \Phi)$ be a triple defined by a non trivial extension of the form (3.6), with F locally free. If E_2 and F are stable then T is α-stable for $\alpha_L < \alpha < \alpha_M$.

From this, we have the following result (Theorem 7.7 in [8]).

Theorem 3.17. *Let $n_1 > n_2$, $d_1/n_1 > d_2/n_2$, $(n_1 - n_2, d_1 - d_2) = 1$ and $(n_2, d_2) = 1$. Then the moduli space $\mathcal{N}^s_{\alpha_M^-}(n_1, n_2, d_1, d_2)$ is smooth of dimension $n_2 d_1 - n_1 d_2 + 1$ and it is isomorphic to a $\mathbb{P}^N$-fibration over $\mathcal{M}_C(n_2, d_2) \times \mathcal{M}_C(n_1 - n_2, d_1 - d_2)$, whose fibre over the point (E_2, F) is given by $\mathbb{P}H^1(C, E_2 \otimes F^*)$, and $N = n_2 d_1 - n_1 d_2 - 1$.*

Remark 3.18. The case $n_1 < n_2$ reduces to the situation in Theorem 3.17 by considering the dual triple.

We prove now that the Fourier–Mukai transform preserves stability for "large" values of the parameter α.

Theorem 3.19. *Let $T = (E_1, E_2, \Phi)$ be an α-stable triple such that $(n_1 - n_2, d_1 - d_2) = 1$, $(n_2, d_2) = 1$, $n_1 \neq n_2$ and $\alpha_L < \alpha < \alpha_M$. Suppose also that $d_1 > 0$, $d_2 > 0$ and $d_1 - d_2 > 0$ (respectively $d_1 < 0$, $d_2 < 0$ and $d_1 - d_2 < 0$); then T is IT_0 (respectively IT_1) and the transformed triple $\hat{T} = (\hat{E}_1, \hat{E}_2, \hat{\Phi})$ is $\hat{\alpha}$-stable for $\hat{\alpha} \in (\hat{\alpha}_L, \hat{\alpha}_M)$ where $\hat{\alpha}_L$ and $\hat{\alpha}_M$ are the values corresponding to the transformed triple $\hat{T}$.*

Proof. We prove the IT_0 case, the proof of the other case is entirely similar. Without loss of generality we may assume $n_1 > n_2$. By Proposition 3.16 the map $\Phi : E_2 \to E_1$ is injective and the quotient sheaf F in $0 \to E_2 \to E_1 \to F \to 0$ is locally free. Moreover, E_2 and F are stable, and hence IT_0, from which it follows that E_1 is IT_0. Transforming the above sequence we get

$$0 \to \hat{E}_2 \to \hat{E}_1 \to \hat{F} \to 0.$$

Since the Fourier–Mukai transform preserves stability (Proposition 2.1) it follows that $\hat{E}_2$ and $\hat{F}$ are stable. By Proposition 3.16 $\hat{T} = (\hat{E}_1, \hat{E}_2, \hat{\Phi})$ is $\hat{\alpha}$-stable for $\hat{\alpha} \in (\hat{\alpha}_L, \hat{\alpha}_M)$. The proof of the converse is identical. □

Under the same conditions of Theorem 3.19, we have the following.

Corollary 3.20. *The Fourier–Mukai transform induces an isomorphism between the moduli spaces of* IT_i *stable triples:*

$$\mathcal{N}^s_{\alpha_M^-}(n_1, n_2, d_1, d_2) \simeq \mathcal{N}^s_{\hat{\alpha}_M^-}((-1)^i d_1, (-1)^i d_2, (-1)^{i+1} n_1, (-1)^{i+1} n_2).$$

As a consequence, the Fourier–Mukai transform yields an isomorphism between the $\mathbb{P}^N$-*fibrations described in* Theorem 3.17.

3.5. Applications to moduli spaces on $C \times \mathbb{P}^1$

One notable application of the theory of triples is the construction of slope-stable bundles on $C \times \mathbb{P}^1$ with respect to the polarization ω_α, with $\alpha > 0$ (see Theorem 9.2 in [8]). It seems quite natural to use the relative transform $\mathcal{S}_{\mathbb{P}^1}$ to further study the properties of those bundles and to produce new examples of stable bundles. We give in this section a result on the preservation of stability for a class of bundles on $C \times \mathbb{P}^1$ which can not be handled using the standard techniques based on choosing “suitable polarizations” [26] as done for example in [9] or [16], because the polarizations ω_α are not suitable; the reason for this being that there exist $SU(2)$-equivariant bundles which are ω_α-stable and whose restriction to a fibre, is never stable (here we are assuming $\Phi \neq 0$). To see this, take E such that $\Phi : E_2 \to E_1$ is not an isomorphism and note that the restriction of such a bundle to a fibre C_t is given by an extension

$$0 \to E_1 \to E_t \to E_2 \to 0.$$

Since the associated triple is stable, Lemma 4.5 in [7] implies that $\mathrm{Ext}^1(E_2, E_1) = 0$ whenever Φ is not an isomorphism, Therefore, the previous extension is always split and the restriction E_t is not stable.

The following Proposition follows now immediately.

Proposition 3.21. *Let $T = (E_1, E_2, \Phi)$ be an α-stable triple and let E be its associated vector bundle on $C \times \mathbb{P}^1$. Then T is* IT_i *if and only if E is* IT_i *with respect to $\mathcal{S}_{\mathbb{P}^1}$.*

We can use this Proposition to prove the following result.

Theorem 3.22. *Let $T = (E_1, E_2, \Phi)$ be an α-stable triple with $E_1 \simeq E_2$ and $\Phi \neq 0$. Assume that either* $\mathrm{rk}(E_1) = \mathrm{rk}(E_2) > 1$ *or* $\deg E_1 = \deg E_2 \neq 0$. *Then the associated $SU(2)$-equivariant bundle E on $C \times \mathbb{P}^1$ is IT and the Fourier–Mukai transform $\hat{E}$ is polystable. Moreover, the triple $\hat{T} = (\hat{E}_1, \hat{E}_2, \hat{\Phi})$ is $\hat{\alpha}$-stable for any $\hat{\alpha} > 0$.*

Proof. Thanks to Theorem 3.4, we have $E \simeq (p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(1)) \oplus (p^*E_2 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(1))$ with $E_1 \simeq E_2$ stable. The base change property for the Fourier–Mukai transform implies

$$\mathcal{S}_{\mathbb{P}^1}(E) \simeq (p^*\mathcal{S}(E_1) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(1)) \oplus (p^*\mathcal{S}(E_2) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(1))$$

Therefore, E is IT if and only if $E_1 \simeq E_2$ is IT with respect to $\mathcal{S}$, and this follows from Proposition 2.1 since the stability of $E_1 \simeq E_2$ implies that its degree is not zero unless the rank is 1. The polystability of $\hat{E}$ is a consequence of the above expression for $\hat{E}$ and the fact that $\mathcal{S}$ preserves stability, see Proposition 2.1.

On the other hand, let us recall that a triple (E_1, E_2, Φ) with $E_1 \simeq E_2$ is α-stable, for any $\alpha > 0$, if and only if Φ is an isomorphism and $E_1 \simeq E_2$ is stable [7, Lemma 4.6]. These conditions are preserved by $\mathcal{S}$, therefore $\hat{T}$ is $\hat{\alpha}$-stable for any $\hat{\alpha} > 0$. □

Collecting previous results, particularly Theorem 3.19, Theorems 3.14 and 3.4, we have.

Theorem 3.23. *Let T be an α-stable triple of type (n_1, n_2, d_1, d_2) with α and (n_i, d_i) satisfying one of the conditions*

(i) $(n_1 - n_2, d_1 - d_2) = 1$, $(n_2, d_2) = 1$, $n_1 \neq n_2$ *and* $\alpha_L < \alpha < \alpha_M$. *Suppose also that* $d_i > 0$ *for* $i = 1, 2$, $d_1 - d_2 > 0$ (*respectively* $d_i < 0$ $i = 1, 2$, $d_1 - d_2 < 0$) *and* $\alpha_L < \alpha < \alpha_M$,
(ii) $(n_1, d_1) = 1$, $(n_2, d_2) = 1$, $d_1 d_2 > 0$ *and* $\alpha_m < \alpha < \alpha_m^+$,

(*i.e. one of the conditions in* Theorems 3.19 or 3.14). *Then, the corresponding* $SU(2)$-*equivariant bundle* E *on* $C \times \mathbb{P}^1$ *is* IT. *Moreover if* E_1 *and* E_2 *are not isomorphic, then the Fourier–Mukai transform* $\hat{E}$ *is stable with respect to the polarization* $\omega_{\hat{\alpha}}$, *where* $\hat{\alpha}$ *is the corresponding parameter for the transformed triple according to* Theorem 3.19 *in case* (i) *and to* Theorem 3.14 *in case* (ii).

The relative Fourier–Mukai transform induces an isomorphism between the corresponding moduli spaces of $SU(2)$-equivariant bundles as follows from the previous Theorem and Remarks 3.5 and 3.11. Therefore we have:

Corollary 3.24. *Let* $\mathcal{N}_\alpha^s$ *be a moduli space of* α-*stable triple satisfying one of the conditions* (i) *or* (ii). *Let* $\mathcal{M}_\alpha^{SU(2)}$ *be the corresponding moduli space of* $SU(2)$-*equivariant bundles on* $C \times \mathbb{P}^1$. *Then the relative Fourier–Mukai transform gives an isomorphism*

$$\mathcal{S}_{\mathbb{P}^1} : \mathcal{M}_\alpha^{SU(2)} \xrightarrow{\sim} \mathcal{M}_{\hat{\alpha}}^{SU(2)}.$$

4. Nahm transforms for triples

4.1. Relative Nahm transform

In this section, we modify the absolute Nahm transform to produce a relative version of it.

For every elliptic curve C we consider the projections $q : X = C \times \mathbb{P}^1 \to \mathbb{P}^1$, $\hat{q} : \hat{X} = \hat{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ where $\hat{C}$ is the dual elliptic curve. We endow the pull-back $\mathcal{P}_{\mathbb{P}^1}$ of the Poincaré line bundle to $X \times_{\mathbb{P}^1} \hat{X}$, with the pull-back connection $\nabla_{\mathcal{P}_{\mathbb{P}^1}}$. For every point $\hat{x} = (\xi, t) \in \hat{C} \times \mathbb{P}^1$ we endow the Hermitian line bundle $\mathcal{P}_{\mathbb{P}^1,\hat{x}} \equiv \mathcal{P}_{\mathbb{P}^1|X_{\hat{q}(\hat{x})}} \to X_{\hat{q}(\hat{x})}$, obtained by restricting $\mathcal{P}_{\mathbb{P}^1}$ to the fiber $X_{\hat{q}(\hat{x})} \subset X \times_{\mathbb{P}^1} \hat{X}$ of q over $\hat{q}(\hat{x}) \in \mathbb{P}^1$, with the flat unitary connection $\nabla_{\hat{x}}$ given by the restriction of $\nabla_{\mathcal{P}_{\mathbb{P}^1}}$. In this way $\hat{X}$ parametrizes the gauge equivalence classes of Hermitian flat line bundles along the fibers of $q : X \to \mathbb{P}^1$.

Let us consider a Hermitian vector bundle $E \to X$ with a unitary connection ∇. We denote by E_t the restriction of E to the fibre $X_t = q^{-1}(t)$, ∇_t is the restriction of ∇ to $\underline{E}_t$. On the vector bundle $E_{\hat{q}(\hat{x})} \otimes \mathcal{P}_{\mathbb{P}^1,\hat{x}}$, we have the connection $\nabla_{\hat{x}} = \nabla_{\hat{q}(\hat{x})} \otimes 1 + 1 \otimes \overline{\nabla}_{\hat{x}}$. Therefore, we have the family of coupled Dirac operators

$$D_{\hat{x}} = \sqrt{2}\bar{\partial}^{*}_{E_{\hat{q}(\hat{x})}\otimes\mathcal{P}_{\mathbb{P}^1,\hat{x}}} : \Omega^0(X_{\hat{q}(\hat{x})}, E_{\hat{q}(\hat{x})} \otimes \mathcal{P}_{\mathbb{P}^1,\hat{x}}) \to \Omega^{0,1}(X_{\hat{q}(\hat{x})}, E_{\hat{q}(\hat{x})} \otimes \mathcal{P}_{\mathbb{P}^1,\hat{x}}).$$

As in the absolute case we define the index $\mathrm{Ind}(D)$ of this family of Dirac operators D parametrized by $\hat{C} \times \mathbb{P}^1$. The relative Nahm transform maps a Hermitian vector bundle with a unitary connection over $C \times \mathbb{P}^1$ into a Hermitian vector bundle with a unitary connection over $\hat{C} \times \mathbb{P}^1$.

Definition 4.1. Let (E, ∇) be a pair formed by a Hermitian vector bundle E over $C \times \mathbb{P}^1$ and a unitary connection ∇ on E. We say that (E, ∇) is an $\mathrm{IT}_{\mathbb{P}^1}$ (index Theorem) pair relative to $\mathbb{P}^1$ if either $\mathrm{Coker}\, D = 0$ or $\mathrm{Ker}\, D = 0$. In the first case we say that (E, ∇) is an $\mathrm{IT}_{\mathbb{P}^1,0}$-pair, whereas in the second we call it an $\mathrm{IT}_{\mathbb{P}^1,1}$ pair. The transformed bundle of an $\mathrm{IT}_{\mathbb{P}^1,i}$-pair is, according to the parity of i, the vector bundle $\hat{E} = \pm\mathrm{Ind}(D) \to \hat{C} \times \mathbb{P}^1$.

Proceeding in the same way as in the absolute case we can endow the transformed vector bundle of an $\mathrm{IT}_{\mathbb{P}^1}$-pair with a Hermitian metric and a unitary connection in a natural way. In doing this, since all the fibrations involved are trivial, the main difference one encounters is that the parameter space of the family is enlarged from $\hat{C}$ to $\hat{C} \times \mathbb{P}^1$, but since $X_{\hat{q}(\hat{x})} \simeq C$ the Dirac operators are still defined on vector bundles over the elliptic curve C. Therefore, the theory parallels the one developed in the absolute setting.

Definition 4.2. Let (E, ∇) be an $\mathrm{IT}_{\mathbb{P}^1}$-pair. We call $(\hat{E}, \hat{\nabla})$ the relative Nahm transform of (E, ∇) and denote it by $\mathcal{N}_{\mathbb{P}^1}(E, \nabla)$.

Let $E \to C \times \mathbb{P}^1$ be a holomorphic vector bundle endowed with a unitary connection ∇ compatible with the holomorphic structure. Since the spin^c Dirac operator $D_{\hat{x}}$ gets identified with the Dolbeault–Dirac operator of $E_{\hat{q}(\hat{x})} \otimes \mathcal{P}_{\mathbb{P}^1,\hat{x}}$, by Hodge theory and the Dolbeault isomorphism, we have

$$\mathrm{Ker}\, D_{\hat{x}} \simeq H^0(X_{\hat{q}(\hat{x})}, E_{\hat{q}(\hat{x})} \otimes \mathcal{P}_{\mathbb{P}^1,\hat{x}}) \tag{4.1}$$

$$\mathrm{Coker}\, D_{\hat{x}} \simeq H^1(X_{\hat{q}(\hat{x})}, E_{\hat{q}(\hat{x})} \otimes \mathcal{P}_{\mathbb{P}^1,\hat{x}}). \tag{4.2}$$

Let us suppose that E is IT_i with respect to the relative Fourier–Mukai transform described in Section 3.2. The isomorphisms (4.1) and (4.2) mean that (E, ∇) is an $\mathrm{IT}_{\mathbb{P}^1,i}$-pair. As we saw there, by [2, Theorem 2] or [12, Theorem 3.2.8], we have a natural C^∞ vector bundle isomorphism induced by Hodge theory

$$\phi_{\mathbb{P}^1} : \mathcal{S}^i_{\mathbb{P}^1}(E) \tilde{\to} \hat{E}.$$

Moreover, since the Poincaré bundle $\mathcal{P}_{\mathbb{P}^1} \to C \times \hat{C}$ is a holomorphic line bundle and the connection $\nabla_{\mathcal{P}_{\mathbb{P}^1}}$ is compatible with the holomorphic structure, the same arguments that in the absolute case led us to prove Theorem 2.12 give us now the following.

Theorem 4.3. *Let F_1, F_2 be Hermitian vector bundles over $C \times \mathbb{P}^1$ endowed with unitary connections ∇_1, ∇_2 such that (F_1, ∇_1), (F_2, ∇_2) are $\mathrm{IT}_{\mathbb{P}^1,i}$-pairs with respect to the Nahm transform. Then we have*

1. *The connections $\hat{\nabla}_1$, $\hat{\nabla}_2$ are compatible with the holomorphic structures of $\mathcal{S}^i_{\mathbb{P}^1}(F_1)$, $\mathcal{S}^i_{\mathbb{P}^1}(F_2)$, respectively. Thus, the curvature of the connections $\hat{\nabla}_1$, $\hat{\nabla}_2$ is of type $(1, 1)$.*
2. *For every holomorphic morphism $\Psi : F_1 \to F_2$ we have an induced holomorphic morphism $\mathcal{N}(\Phi) : \hat{F}_1 \to \hat{F}_2$ and a commutative diagram*

$$\begin{tikzcd} \widehat{F_1} \arrow[r, "\phi_{F_1}"] \arrow[d, "\mathcal{N}_{\mathbb{P}^1}(\Psi)"'] & \mathcal{S}(F_1) \arrow[d, "\mathcal{S}_{\mathbb{P}^1}(\Psi)"] \\ \widehat{F_2} \arrow[r, "\phi_{F_2}"] & \mathcal{S}(F_2) \end{tikzcd} \tag{4.3}$$

4.2. Relative Nahm transform for $SU(2)$-invariant Einstein–Hermitian connections

Let us suppose that E_1, E_2 are complex Hermitian vector bundles over C and let us choose an $SU(2)$-invariant metric on $H^{\otimes 2}$. We put on the bundle $E = p^*E_1 \oplus (p^*E_2 \otimes q^*H^{\otimes 2})$ the Hermitian metric which is determined in a natural way by the Hermitian metrics of E_1, E_2 and $H^{\otimes 2}$.

By Proposition 3.5 in [14] there is a one to one correspondence between the $SU(2)$-invariant unitary connections on E and the triples $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ formed by unitary connections ∇_1, ∇_2 on E_1, E_2, respectively, and a C^∞ vector bundle morphism $\Phi : E_2 \to E_1$. Moreover, this correspondence also holds at the level of $SU(2)$-invariant holomorphic structures on E. Before discussing it we introduce the following.

Definition 4.4. We call a triple $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ integrable if $\Phi : E_2 \to E_1$ is holomorphic with respect to the holomorphic structures determined by the connections ∇_1 and ∇_2.

Proposition 3.9 in [14] gives us a one to one correspondence between $SU(2)$-invariant holomorphic structures on E, considered as integrable $SU(2)$-invariant connections (i.e. connections with curvature of type $(1, 1)$), and integrable triples $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$.

This is precisely the content of Proposition 3.1 which gives us a bijective correspondence between the $SU(2)$-invariant holomorphic structures on E and holomorphic triples $T = (\mathcal{E}_1 = (E_1, \bar\partial^{\nabla_1}), \mathcal{E}_2 = (E_2, \bar\partial^{\nabla_2}), \Phi)$.

Let us denote by $\nabla^{\mathcal{T}}$ the $SU(2)$-invariant integrable connection on E determined by an integrable triple $\mathcal{T}$. If we express its curvature with respect to the splitting $E = p^*E_1 \oplus (p^*E_2 \otimes q^*H^2)$, we have

$$F^{\nabla^{\mathcal{T}}} = \begin{pmatrix} p^*F^{\nabla_1} - \beta \wedge \beta^* & \partial\beta \\ -\bar\partial\beta^* & p^*F^{\nabla_2} \otimes 1 + 1 \otimes q^*F^{\nabla'} - \beta^* \wedge \beta, \end{pmatrix} \tag{4.4}$$

where F^{∇_i} is the curvature of the connection ∇_i, $F^{\nabla'}$ is the curvature of the unique $SU(2)$-invariant unitary connection on $H^{\otimes 2}$, $\beta = p^*\Phi \otimes q^*\eta$, with η an $SU(2)$-invariant section of $H^{\otimes -2}$ and $\bar\partial$ is the Cauchy-Riemann operator determined by the connections ∇_1, ∇_2 and ∇', for further details see [14,7].

We want to study the relative Nahm transform of the $SU(2)$-equivariant bundles $(E, \nabla^{\mathcal{T}})$ associated to integrable triples.

The following is straightforward.

Proposition 4.5. *Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be an integrable triple on C and let $(E, \nabla^{\mathcal{T}})$ be its associated bundle with connection over $C \times \mathbb{P}^1$. If both (E_1, ∇_1), (E_2, ∇_2) are IT_i-pairs then $(E, \nabla^{\mathcal{T}})$ is an $\mathrm{IT}_{\mathbb{P}^1, i}$ pair.*

Given an integrable triple $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ such that (E_1, ∇_1) and (E_2, ∇_2) are IT_i-pairs we can form the triple $\hat{\mathcal{T}} = ((\hat E_1, \hat\nabla_1), (\hat E_2, \hat\nabla_2), \hat\Phi)$ obtained by means of the absolute Nahm transform. Here we have denoted by $\hat\Phi$ the Nahm transform $\mathcal{N}(\Phi)$. By the sake of brevity the same notation is used hereafter. On the other hand, if $(E, \nabla^{\mathcal{T}})$ is the vector bundle with connection over $C \times \mathbb{P}^1$ associated to the triple $\mathcal{T}$, we can apply to it the relative Nahm transform to obtain $\mathcal{N}_{\mathbb{P}^1}(E, \nabla^{\mathcal{T}})$. Taking into account the compatibility between the Fourier–Mukai and Nahm transforms, Theorems 2.12 and 4.3 and Proposition 3.10 we have.

Proposition 4.6. *$\mathcal{N}_{\mathbb{P}^1}(E, \nabla^{\mathcal{T}})$ is the vector bundle on $\hat C \times \mathbb{P}^1$ associated to the triple $\hat{\mathcal{T}} = ((\hat E_1, \hat\nabla_1), (\hat E_2, \hat\nabla_2), \hat\Phi)$.*

Definition 4.7. Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be an integrable triple on C. We say that it satisfies the τ-coupled vortex equations if

$$\begin{aligned} i\Lambda F^{\nabla_1} + \Phi\Phi^* &= 2\pi\tau \mathrm{Id}_{E_1} \\ i\Lambda F^{\nabla_2} - \Phi^*\Phi &= 2\pi\tau' \mathrm{Id}_{E_2}, \end{aligned}$$

Note that in order to have solutions τ, τ' must fulfill the following equation

$$n_1\tau + n_2\tau' = d_1 + d_2, \tag{4.5}$$

with $n_i = \mathrm{rank}(E_i)$ and $d_i = \deg(E_i)$.

The following Proposition was proved in [13] (see also [7]).

Proposition 4.8. *Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be an integrable triple and let $\nabla^{\mathcal{T}}$ be the corresponding connection on E. Let τ and τ' be related by (4.5) and let us suppose that*

$$\alpha = \frac{(n_1 + n_2)\tau - d_1 - d_2}{n_2} > 0.$$

Then $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ satisfies the τ-coupled vortex equations if and only if $\nabla^{\mathcal{T}}$ is a Einstein–Hermitian connection on $E \to C \times \mathbb{P}^1$ with respect to the Kähler form $\omega_\alpha = \alpha/2\, p^*\omega_C + q^*\omega_{\mathbb{P}^1}$, where $\omega_{\mathbb{P}^1}$ is the Fubini-Study Kähler form normalized to volume one and ω_C is a Kähler form of unit volume.

Proposition 4.9. *Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be an integrable triple on C which satisfies the τ-coupled vortex equations and let $X = C \times \mathbb{P}^1$. Then:*

(i) *If the Hermitian endomorphisms $2\pi\tau \mathrm{Id}_{E_1} - \Phi\Phi^*$ and $2\pi\tau' \mathrm{Id}_{E_2} + \Phi^*\Phi$ are non-negative and there exist $x_1, x_2 \in C$ such that $2\pi\tau \mathrm{Id}_{E_1} - \Phi\Phi^*(x_1) > 0$, $2\pi\tau' \mathrm{Id}_{E_2} + \Phi^*\Phi(x_2) > 0$, then $(E, \nabla^{\mathcal{T}})$ is an $\mathrm{IT}_{\mathbb{P}^1,0}$-pair and (E_1, ∇_1), (E_2, ∇_2) are IT_0 pairs.*

(ii) *If the Hermitian endomorphisms $2\pi\tau \mathrm{Id}_{E_1} - \Phi\Phi^*$ and $2\pi\tau' \mathrm{Id}_{E_2} + \Phi^*\Phi$ are non-positive and there exist $x_1, x_2 \in C$ such that $2\pi\tau \mathrm{Id}_{E_1} - \Phi\Phi^*(x_1) < 0$, $\pi\tau' \mathrm{Id}_{E_2} + \Phi^*\Phi(x_2) < 0$, then $(E, \nabla^{\mathcal{T}})$ is an $\mathrm{IT}_{\mathbb{P}^1,1}$-pair and (E_1, ∇_1), (E_2, ∇_2) are IT_1 pairs.*

Proof. For every $\hat{x} = (\xi, t)$ the restriction of $E = p^*E_1 \oplus (p^*E_2 \otimes q^*H^{\otimes 2})$ to $X_{\hat{q}(\hat{x})} \simeq C$ is isomorphic to $E_1 \oplus E_2$ as C^∞ bundles. Now (4.4) implies that the curvature of $\nabla_{\hat{q}(\hat{x})}$ with respect to the splitting $E_{\hat{q}(\hat{x})} \simeq E_1 \oplus E_2$ is

$$F^{\nabla_{\hat{q}(\hat{x})}} = \begin{pmatrix} F^{\nabla_1} & 0 \\ 0 & F^{\nabla_2} \end{pmatrix}.$$

The claim now follows from Theorem 2.6. □

4.3. Covariantly constant triples

Definition 4.10. Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be an integrable triple on C. We will say that $\mathcal{T}$ is covariantly constant if $\Phi\Phi^*$ is covariantly constant with respect to ∇_1 and $\Phi^*\Phi$ is covariantly constant with respect to ∇_2.

Remark 4.11. Denote by ∇ the connection naturally induced on $\mathrm{Hom}(E_2, E_1)$ by ∇_1 and ∇_2. If Φ is covariantly constant with respect to ∇ then it is easy to check that $\mathcal{T}$ is covariantly constant. Moreover, Φ is covariantly constant with respect to ∇ if and only if $\Phi : E_2 \to E_1$ is an anti-holomorphic map.

Proposition 4.12. *Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be a covariantly constant integrable triple on C. Then, we have holomorphic orthogonal decompositions*

$$E_1 \simeq \operatorname{Ker}\Phi^* \oplus E_1'$$
$$E_2 \simeq \operatorname{Ker}\Phi \oplus E_2'$$

which are compatible with the connections, and Φ induces an holomorphic isomorphism $\Phi : E_2' \to E_1'$.

Proof. Since $\Phi\Phi^*$ and $\Phi^*\Phi$ are covariantly constant vector bundle endomorphisms, they are holomorphic and their eigenvalues are constant. Moreover, $\Phi\Phi^*$, $\Phi^*\Phi$ are positive Hermitian endomorphisms whose spectrum may differ only at 0; Therefore, we have orthogonal decompositions

$$E_1 = \operatorname{Ker}\Phi^* \oplus E_1(\lambda_1) \oplus \ldots \oplus E_1(\lambda_k)$$
$$E_2 = \operatorname{Ker}\Phi \oplus E_2(\lambda_1) \oplus \ldots \oplus E_2(\lambda_k),$$

where $E_1(\lambda_i)$, $E_2(\lambda_i)$ are the eigenbundles with eigenvalue $\lambda_i \neq 0$ with respect to the holomorphic endomorphisms $\Phi\Phi^*$ and $\Phi^*\Phi$, respectively. Since these endomorphisms are covariantly constant, the subbundles $E_1(\lambda_i)$, $E_2(\lambda_i)$ are preserved by the connections ∇_1, ∇_2, respectively. Moreover, for every λ_i we have an isomorphism

$$\Phi : E_2(\lambda_i) \xrightarrow{\sim} E_1(\lambda_i)$$

Therefore, if we denote $E_1' = E_1(\lambda_1) \oplus \ldots \oplus E_1(\lambda_k)$, $E_2' = E_2(\lambda_1) \oplus \ldots \oplus E_2(\lambda_k)$, we have an isomorphism

$$\Phi : E_2' \xrightarrow{\sim} E_1'$$

as required. □

With the same notations as above, we have the following

Proposition 4.13. *Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be a covariantly constant integrable triple on C. Then $\mathcal{T}$ satisfies the τ-coupled equations if and only if*

1. ∇_1 *induces a constant central curvature connection on* $\operatorname{Ker}\Phi^*$ *with factor* $2\pi\tau$, *unless* $\operatorname{Ker}\Phi^* = 0$, *and a constant central curvature connection on* E_1' *with factor* $\pi(\tau + \tau')$ *unless* $E_1' = 0$.
2. ∇_2 *induces a constant central curvature connection on* $\operatorname{Ker}\Phi$ *with factor* $2\pi\tau'$ *unless* $\operatorname{Ker}\Phi = 0$ *and a constant central curvature connection on* E_2' *with factor* $\pi(\tau + \tau')$ *unless* $E_2' = 0$.

Proof. Since $\mathcal{T}$ is covariantly constant we have the decompositions

$$E_1 = \mathrm{Ker}\,\Phi^* \oplus E_1(\lambda_1) \oplus \ldots \oplus E_1(\lambda_k)$$
$$E_2 = \mathrm{Ker}\,\Phi \oplus E_2(\lambda_1) \oplus \ldots \oplus E_2(\lambda_k),$$

provided by Proposition 4.12. Moreover, since $\mathcal{T}$ satisfies the τ-coupled equations, we have

$$i\Lambda F^{\nabla_1} = 2\pi\tau \mathrm{Id}_{E_1} - \Phi\Phi^*$$
$$i\Lambda F^{\nabla_2} = 2\pi\tau' \mathrm{Id}_{E_2} + \Phi^*\Phi.$$

Therefore, we have

$$i\Lambda F^{\nabla_1}{}_{|\mathrm{Ker}\,\Phi^*} = 2\pi\tau \mathrm{Id}_{\mathrm{Ker}\,\Phi^*} \quad i\Lambda F^{\nabla_1}{}_{|E_1(\lambda_i)} = (2\pi\tau - \lambda_i)\mathrm{Id}_{E_1(\lambda_i)} \tag{4.6}$$

$$i\Lambda F^{\nabla_2}{}_{|\mathrm{Ker}\,\Phi} = 2\pi\tau' \mathrm{Id}_{\mathrm{Ker}\,\Phi} \quad i\Lambda F^{\nabla_2}{}_{|E_2(\lambda_i)} = (2\pi\tau' + \lambda_i)\mathrm{Id}_{E_2(\lambda_i)}. \tag{4.7}$$

This implies that $\mathrm{Ker}\,\Phi^*$, $E_1(\lambda_i)$, $\mathrm{Ker}\,\Phi$, $E_2(\lambda_i)$ are bundles with constant central curvature connection with slopes

$$\mu(\mathrm{Ker}\,\Phi^*) = \tau \quad \mu(E_1(\lambda_i)) = \tau - \frac{\lambda_i}{2\pi}$$
$$\mu(\mathrm{Ker}\,\Phi) = \tau' \quad \mu(E_2(\lambda_i)) = \tau' + \frac{\lambda_i}{2\pi}.$$

But since $E_1(\lambda_i)$ is isomorphic to $E_2(\lambda_i)$ we must have $\mu(E_1(\lambda_i)) = \mu(E_2(\lambda_i))$, that is $\lambda_i = \pi(\tau - \tau')$. Therefore, in the above decompositions there is only one eigenvalue and if we substitute λ_i in (4.6) and (4.7) we get the required values for the factors of the constant central curvature connections. The converse statement is just a simple checking. □

Corollary 4.14. *Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be a covariantly constant integrable triple on C which satisfies the τ-coupled equations. Then $\mathcal{T}$ is $(\tau - \tau')$-polystable. Moreover, $\mathcal{T}$ decomposes as a sum of $(\tau - \tau')$-polystable triples.*

$$\mathcal{T} = (\mathrm{Ker}\,\Phi^*, 0, 0) \oplus (E_1', E_2', \Phi) \oplus (0, \mathrm{Ker}\,\Phi, 0).$$

Proof. The Hitchin–Kobayashi correspondence for triples, [7, Theorem 5.1], establishes an equivalence between triples that satisfy the τ-coupled vortex equations and $(\tau - \tau')$-polystable triples. Therefore, the Corollary follows at once.

However, in the present case it is possible to give a direct proof. Since $\mathcal{T}$ satisfies the τ-coupled vortex equations, Proposition 4.13 implies that $\mathrm{Ker}\,\Phi^*$, $E_1' \simeq E_2'$ and $\mathrm{Ker}\,\Phi$ are polystable bundles with slopes

$$\mu(\mathrm{Ker}\,\Phi^*) = \tau, \qquad \mu(E_1') = \mu(E_2') = \tfrac{1}{2}(\tau + \tau'), \qquad \mu(\mathrm{Ker}\,\Phi) = \tau'.$$

Therefore, we have

$$\mu_\alpha(\operatorname{Ker}\Phi^*, 0, 0) = \mu(\operatorname{Ker}\Phi^*) = \tau$$
$$\mu_\alpha(E_1', E_2', \Phi) = \mu(E_2') = \mu(E_1') + \tfrac{\alpha}{2} = \tau$$
$$\mu_\alpha(0, \operatorname{Ker}\Phi, 0) = \mu(\operatorname{Ker}\Phi) + \alpha = \tau,$$

where $\alpha = (\tau - \tau')$. Since E_2' carries a constant central curvature, there exists an orthogonal decomposition

$$E_2' = E_2^{(1)} \oplus \ldots \oplus E_2^{(m)}$$

compatible with the connection and such that every factor carries an irreducible constant central curvature connection. Since $\Phi^*\Phi = \lambda\, \mathrm{Id}_{E_2}$ it follows that we have an orthogonal decomposition

$$E_1' = \Phi(E_2^{(1)}) \oplus \ldots \oplus \Phi(E_2^{(m)})$$

Thus, the triple $(E_1', E_2', \Phi_{|E_2'})$ splits into the direct sum of subtriples $(E_2^{(i)}, \Phi(E_2^{(i)}), \Phi_{|E_2^{(i)}})$ with $E_2^{(i)}$ stable and $\Phi_{|E_2^{(i)}}$ an isomorphism. By [7, Proposition 3.21] this implies that $(E_1', E_2', \Phi_{|E_2'})$ is α-polystable. Therefore, $\mathcal{T}$ is α-polystable (see [7, Definition 3.15]). □

Remark 4.15. If $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ is a covariantly constant integrable triple on C which is α-stable with $E_1 \neq 0$ and $E_2 \neq 0$, then the previous Corollary implies that Φ has to be an isomorphism.

As a consequence of Proposition 4.9 we immediately obtain.

Lemma 4.16. *Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be a covariantly constant integrable triple on C which satisfies the τ-coupled equations.*

(i) *If $\tau > 0$ and $\tau' > 0$ then $(E, \nabla^{\mathcal{T}})$ is an $\mathrm{IT}_{\mathbb{P}^1,0}$-pair and $(E_1, \nabla_1), (E_2, \nabla_2)$ are IT_0 pairs.*
(ii) *If $\tau < 0$ and $\tau' < 0$ then $(E, \nabla^{\mathcal{T}})$ is an $\mathrm{IT}_{\mathbb{P}^1,1}$-pair and $(E_1, \nabla_1), (E_2, \nabla_2)$ are IT_1 pairs.*

Theorem 4.17. *Let $\mathcal{T} = ((E_1, \nabla_1), (E_2, \nabla_2), \Phi)$ be a covariantly constant integrable triple on C which satisfies the τ-coupled equations and let $(E, \nabla^{\mathcal{T}})$ be its associated bundle with connection over $C \times \mathbb{P}^1$.*

1. *If $\tau > 0$ and $\tau' > 0$ then the Nahm transform $\hat{\mathcal{T}} = ((\hat{E}_1, \hat{\nabla}_1), (\hat{E}_2, \hat{\nabla}_2), \hat{\Phi})$ is a covariantly constant integrable triple. Moreover, $\hat{\mathcal{T}}$ satisfies the $\hat{\tau}$-coupled equations, for some value of $\hat{\tau}$, if and only if $\tau = \tau'$.*
2. *If $\tau < 0$ and $\tau' < 0$ then the Nahm transform $\hat{\mathcal{T}} = ((\hat{E}_1, \hat{\nabla}_1), (\hat{E}_2, \hat{\nabla}_2), \hat{\Phi})$ is a covariantly constant integrable triple. Moreover, $\hat{\mathcal{T}}$ satisfies the $\hat{\tau}$-coupled equations, for some value of $\hat{\tau}$, if and only if $\tau = \tau'$.*

Proof. Proposition 4.12 gives us a decomposition

$$E_1 \simeq \operatorname{Ker}\Phi^* \oplus E_1'$$
$$E_2 \simeq \operatorname{Ker}\Phi \oplus E_2'$$

Since $\mathcal{T}$ satisfies the τ-couple vortex equations, Proposition 4.13 implies that $(\operatorname{Ker}\Phi^*, \nabla_1)$, $(E_1', \nabla_1) \simeq (E_2', \nabla_2)$ and $(\operatorname{Ker}\Phi, \nabla_2)$ are bundles with constant central curvature with slopes $\mu(\operatorname{Ker}\Phi^*) = \tau$, $\mu(E_1') = \mu(E_2') = \frac{1}{2}(\tau + \tau')$, $\mu(\operatorname{Ker}\Phi) = \tau'$. Now if we apply the Nahm transform and denote $(\hat{\Phi})^*$ by $\hat{\Phi}^*$, Theorem 2.11 implies that $(\operatorname{Ker}\hat{\Phi}^* = \widehat{\operatorname{Ker}}\,\Phi^*, \hat{\nabla}_1)$, $(\hat{E}_1', \hat{\nabla}_1) \simeq (\hat{E}_2', \hat{\nabla}_2)$ and $(\operatorname{Ker}\hat{\Phi} = \widehat{\operatorname{Ker}}\,\Phi, \hat{\nabla}_2)$ are bundles with constant central curvature and we get a decomposition

$$\hat{E}_1 \simeq \operatorname{Ker}\hat{\Phi}^* \oplus \hat{E}_1'$$
$$\hat{E}_2 \simeq \operatorname{Ker}\hat{\Phi} \oplus \hat{E}_2'$$

The conditions $(\Phi^*\Phi)_{|E_2'} = \lambda\,\mathrm{Id}_{E_2'}$, $(\Phi\Phi^*)_{|E_1'} = \lambda\,\mathrm{Id}_{E_1'}$ with $\lambda \neq 0$ imply $(\hat{\Phi}^*\hat{\Phi})_{|\hat{E}_2'} = \lambda\,\mathrm{Id}_{\hat{E}_2'}$, $(\hat{\Phi}\hat{\Phi}^*)_{|\hat{E}_1'} = \lambda\,\mathrm{Id}_{\hat{E}_1'}$. Let us prove the first equality in the IT_0 case. Given $s, t \in \hat{E}_{2,\xi} = \operatorname{Ker}\bar{\partial}^{\nabla_{2,\xi}} \subset \Omega^0(E_{2,\xi})$ one has

$$\langle \hat{\Phi}^*\hat{\Phi}(s), t\rangle_{\hat{E}_{2,\xi}} = \langle \hat{\Phi}(s), \hat{\Phi}(t)\rangle_{\hat{E}_{2,\xi}}$$

Taking into account the definition of the Hermitian metric on $\hat{E}_{2,\xi}$ given in (2.3) of Section 2.2 and the definition of $\hat{\Phi}$ given in Theorem 2.12 we get

$$\langle \hat{\Phi}(s), \hat{\Phi}(t)\rangle_{\hat{E}_{2,\xi}} = \int_{C_\xi} \langle \Phi(s), \Phi(t)\rangle_{E_2}\,\omega = \int_{C_\xi} \langle \Phi^*\Phi(s), t\rangle_{E_2}\,\omega$$

Therefore, if $s, t \in \hat{E}_{2,\xi}'$ one has

$$\langle \hat{\Phi}^*\hat{\Phi}(s), t\rangle_{\hat{E}_{2,\xi}} = \lambda\,\langle s, t\rangle_{\hat{E}_{2,\xi}}$$

which proves our claim. The second equality follows in the same way. The proofs in the IT_1 case are entirely similar.

This proves that $((\hat{E}_1, \hat{\nabla}_1), (\hat{E}_2, \hat{\nabla}_2), \hat{\Phi})$ is a covariantly constant integrable triple. Moreover, the slopes of these bundles are $\mu(\operatorname{Ker}\hat{\Phi}^*) = -1/\tau$, $\mu(\hat{E}_1') = \mu(\hat{E}_2') = -2/(\tau + \tau')$, $\mu(\operatorname{Ker}\hat{\Phi}) = -1/\tau'$. An easy computation shows now that $\hat{\mathcal{T}}$ fulfills the conditions of Proposition 4.13 in order to have a solution of the $\hat{\tau}$-coupled vortex equations, for some value of $\hat{\tau}$, if and only if $\tau = \tau'$. $\square$

As a consequence of the preceding Theorem and the Hitchin–Kobayashi correspondence for triples (Theorem 5.1 in [7]), which establishes an equivalence between holomorphic triples which satisfy the τ-coupled equations and α-polystable triples, we have.

Corollary 4.18. *Polystability is not preserved, in general, under the Fourier–Mukai and Nahm transform.*

Proof. It is enough to take any stable bundles F_1, F_2, F such that $\mu(F) = \frac{1}{2}(\mu(F_1) + \mu(F_2))$ and $\mu(F_1) > \mu(F_2)$, which are known to exist since the moduli spaces of stable bundles with fixed coprime rank and degree over an elliptic curve C are isomorphic to C and thus they are not empty (see [27]). Now define the triple $T = (F_1, 0, 0) \oplus (F, F, \mathrm{Id}_F) \oplus (0, F_2, 0)$ and endow F_1, F_2, F with connections of constant curvature compatible with their holomorphic structures according to Donaldson Theorem [11]. Now, Proposition 4.13 implies that T is $(\tau - \tau')$-polystable since, by construction, it satisfies the τ-coupled equations, with $\tau = \mu(F_1)$ and $\tau' = \mu(F_2)$.

If we take $\mu(F_1) \neq \mu(F_2)$, Theorem 4.17 implies that the transformed triple $\hat{T}$ does not satisfy the $\hat{\tau}$-coupled equations for any value of $\hat{\tau}$. By the Hitchin–Kobayashi correspondence for triples [7, Theorem 5.1], this implies that $\hat{T}$ is not polystable. □

The preservation of stability remains as an open question. Notice that in the case of stable triples (E_1, E_2, Φ) with $E_1 \neq 0$ and $E_2 \neq 0$, the condition of being covariantly constant implies that Φ is an isomorphism (Remark 4.15). Now stability is preserved in the conditions of Theorem 3.22.

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References

[1] M.F. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc. 7 (1957) 414–452.
[2] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, Fourier–Mukai transform and index theory, Manuscripta Math. 85 (1994) 141–163.
[3] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, A Fourier–Mukai transform for stable bundles on $K3$ surfaces, J. Reine Angew. Math. 486 (1997) 1–16.
[4] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, A hyper-Kähler Fourier transform, Differential Geom. Appl. 8 (1998) 239–249.
[5] J.M. Bismut, The Atiyah–Singer index for families of Dirac operators: two heat equation proofs, Invent. Math. 83 (1986) 91–151.
[6] P.J. Braam, P. van Baal, Nahm's transformation for instantons, Comm. Math. Phys. 122 (1989) 267–280.

[7] S. Bradlow, O. García-Prada, Stable triples, equivariant bundles and dimensional reduction, Math. Ann. 304 (1996) 225–252.
[8] S. Bradlow, O. García-Prada, P. Gothen, Moduli spaces of holomorphic triples over compact Riemann surfaces, Math. Ann. 328 (2004) 299–351.
[9] T. Bridgeland, Fourier–Mukai transforms for elliptic surfaces, J. reine angew. Math. 498 (1998) 115–133.
[10] M. Dekker, Moduli Spaces of Stable Sheaves on Abelian Surfaces, Ph.D. Thesis, University of Amsterdam, 1997.
[11] S.K. Donaldson, A new proof of a theorem of Narasimhan and Seshadri, J. Differential Geom. 18 (2) (1983) 269–277.
[12] S.K. Donaldson, P.B. Kronheimer, The geometry of four manifolds, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1991.
[13] O. García-Prada, The Geometry of the Vortex Equation, D. Phil. Thesis, Oxford, 1991.
[14] O. García-Prada, Dimensional reduction of stable bundles, vortices and stable pairs, Int. J. Math. 5 (1) (1994) 1–52.
[15] R. Hartshorne, Lecture Notes in Mathematics Residues and duality, Springer-Verlag, Berlin, 1966, p. 20.
[16] D. Hernández Ruipérez, J.M. Muñoz Porras, Stable sheaves on elliptic fibrations, J. Geom. Phys. 43 (2002) 163–183.
[17] N.J. Hitchin, On the construction of monopoles, Comm. Math. Phys. 89 (2) (1983) 145–190.
[18] M. Jardim, Construction of doubly-periodic instantons, Comm. Math. Phys. 216 (2001) 1–15.
[19] M. Jardim, Nahm transform and spectral curves for doubly-periodic instantons, Comm. Math. Phys. 225 (2002) 639–668.
[20] S. Kobayashi, Differential geometry of complex vector bundles, Publications of the Mathematical Society of Japan, 15, Kanô Memorial Lectures, 5, Princeton University Press, Princeton, NJ, Iwanami Shoten, Tokyo, 1987.
[21] A. Maciocia, Gieseker stability and the Fourier–Mukai transform for abelian surfaces, Quart. J. Math. Oxford Ser. (2) 47 (185) (1996) 87–100.
[22] S. Mukai, Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J. 81 (1981) 153–175.
[23] S. Mukai, Fourier functor and its application to the moduli of bundles on an abelian variety, Adv. Pure Math. 10 (1987) 515–550.
[24] W. Nahm, The construction of all self-dual multimonopoles by the ADHM method, Monopoles in quantum field theory (Trieste, 1981), World Science Publishing, Singapore, 1982, pp. 87–94.
[25] M.S. Narasimhan, C.S. Seshadri, Stable and unitary bundles on a compact Riemann surface, Ann. Math. 82 (1965) 540–564.
[26] K. O'Grady, The weight-two Hodge structure of moduli spaces of sheaves on a $K3$ surface, J. Algebraic Geom. 6 (4) (1997) 599–644.
[27] L.W. Tu, Semistable bundles over an elliptic curve, Adv. Math. 98 (1993) 1–26.
[28] K. Yoshioka, Moduli spaces of stable sheaves on abelian surfaces, Math. Ann. 321 (4) (2001) 817–884.